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EARTH PRESSURE, RETAINING WALLS *and* BINS

BY

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FIRST EDITION
FIRST THOUSAND

145090.
29/1/18.

NEW YORK
JOHN WILEY & SONS, Inc.
London: CHAPMAN & HALL, Limited
1916

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Publishers Printing Company
207-217 West Twenty-fifth Street, New York

PREFACE

Over a century ago, Coulomb formulated the laws of friction and cohesion as affecting a mass of earth, and devised the "sliding-wedge" hypothesis to effect the computation of earth thrust against a wall. For some reason—doubtless on account of the complexity of the analysis and lack of experimental determination of the coefficients of cohesion—the theory of earth pressure was subsequently developed by many noted authors (Poncelet, Weyrauch, and others) after Coulomb's hypothesis, but for an earth supposed to be devoid of cohesion.

In 1856, Rankine published his notable theory of earth pressure, deriving it from considerations pertaining to the equilibrium of an infinitesimal wedge of earth in the interior of a mass of homogeneous earth, supposed to have a free plane surface. Again, the earth was supposed to be devoid of cohesion and likewise to be subjected to no other external force but its own weight.

All of these theories strictly pertain to such materials as dry sand, clean gravel, or loose rock, which are practically devoid of cohesion and affected only by friction between the particles. Although ordinary earth in bank is endowed with both cohesion and friction, it was assumed, when this earth was excavated, more or less pulverized, and placed behind a retaining wall, that the cohesion was temporarily destroyed, so that the theory of earth endowed with friction alone—considering the angle of friction as the angle of repose for the loose earth—could be safely employed; since the filling, under the influence of rains, settlement, and cohesive and chemical affinities, would regain by degrees a large part of the cohesion temporarily lost, so that the thrust would ultimately be less than for the pulverized earth. This, of course, tacitly assumes that the coefficient of friction would not be lowered during the consolidation.

For many years, engineers have expressed their dissatisfaction with a theory thus restricted and which, when applied to earth more or less consolidated—especially clay—was so deficient "in the most vital elements existent in fact." It was thus natural that the pendulum should swing back, so that, in very recent years, the treatment of pressures in coherent earth has been based on Coulomb's original laws. Thus in Résal's comprehensive work,* the subject is treated analytically in great generality.

The author was also led, in the course of a discussion of certain experiments on retaining boards backed by earth and an analysis of the pressures exerted on the bracing of trenches, to develop a complete graphical method for finding the pressures in coherent earth.† More recently, Mr. A. L. Bell, M. Inst. C. E., has added to our knowledge of the subject by experiments on clays and an analysis concerning the supporting power of foundations.‡

From all of the experiments that have been made (as given in Chapter I), the laws of Coulomb seem to be approximately verified, but it is evident that extensive experimenting upon every kind of earth is needed to give confidence. Partly from this lack of experimental data, though mainly because the theory of earth devoid of cohesion is strictly applicable to a granular material, as clean, dry sand, gravel or rip-rap, the theory for such earth is fully developed in Chapters II and III, and numerous applications are made in Chapter IV to the design of retaining walls of stone or reinforced concrete.

It will be found that the analysis of Chapters II–IV is more critical and extended than usual. In Chapter III, the discussion by the "ellipse of stress" method leads up to Mohr's "circular diagram of stress," which is afterward used in Chapter V in treating coherent earth. The author first developed the theory for coherent earth by the analytical method, but eventually decided to use the Mohr diagram, because it not only led to the same results, but gave numerical values with much greater facility than the formulas for the general case, where the earth surface is inclined.

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* Poussée des Terres, II, Théorie des Terres Cohérentes.

† "EXPERIMENTS ON RETAINING WALLS AND PRESSURES ON TUNNELS," by Wm. Cain, *Trans. Am. Soc. C. E.*, vol. LXXII (1911).

‡ Bell on "THE LATERAL PRESSURE AND RESISTANCE OF CLAY AND THE SUPPORTING POWER OF CLAY FOUNDATIONS." *Proc. Inst. C. E.*, vol. CXCIX (1914–15), Part *i*.

In this chapter, the subjects of earth pressures in coherent earth, surfaces of rupture, stable slopes, foundations, the thrust against a retaining wall, the bracing of trenches, and the pressures on tunnel linings, are treated; besides, there is added an independent graphical method for evaluating earth thrust.

The theory of deep bins is given in Chapter VI, and the attempt is made there to reach fairly good results in the vexed subject of the thrusts on the walls of shallow bins filled with coal.

The case of stresses in wedge-shaped reinforced-concrete beams finds an approximate solution in Appendix I, in which a number of diagrams are added to facilitate computation.

Finally, in Appendix II, the results of certain experiments on model retaining walls are added, the discussion of which may prove instructive.

It must be borne in mind that the theory of earth pressures has been necessarily developed for a supposed homogeneous earth, so that it is understood that its indications, for an actual earth, must always be supplemented by the practical judgment of the experienced engineer.

WM. CAIN.

CHAPEL HILL, N. C.
Feb. 6, 1916.

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Earth Pressure, Retaining Walls and Bins

CHAPTER I

LAWS OF FRICTION AND COHESION. TABLES, DIRECTION, AND DISTRIBUTION OF STRESS

1. ORDINARY earths generally consist of sand or gravel with an admixture of clay or clayey matter and humus. When the clayey matter is absent, we have nearly clean sand or gravel, endowed with friction but nearly devoid of cohesion. When the sand or gravel is absent, we have nearly pure clay or clayey earth, still possessing friction and, in addition, a large amount of cohesion. Between these extremes there exists every kind of earth, often saturated with water. Thus in bank sand or gravel, which often includes a certain amount of clay, there is a small though appreciable amount of cohesion, and in ordinary earths before excavation a much larger amount.

Such earths will stand for certain heights with a vertical face, as is illustrated in the case of trenches, where unsupported depths as great as 25 feet have been recorded. If the purpose in view is to ascertain such heights or to deal with the bracing of trenches or tunnel linings, then cohesion is a vital element in the computation.

However, if the design of a retaining wall is in question, it is wise to ignore the cohesion of such earths; for when excavated, more or less pulverized and deposited behind the wall, such "fresh" earths are temporarily nearly devoid of cohesion. It is true that as soon as this fresh earth is deposited, cohesive and chemical affinities come into play, which, under the influence of moisture, increase with time, so that eventually the earth may revert to its state in bank. In fact, such filling has been known in time to become so consolidated as actually to shrink away from the wall and relieve it of all earth thrust. Such a state cannot be considered as permanent, however, for the influence of heavy rains, changes of temperature, freezing and thawing, as well as vibrations due to heavy moving loads, may very materially diminish the cohesion, and to a less extent the friction, particularly if the filling is not well drained.

Such excess of water may lubricate the surface of rupture to such an extent as to lead to thrusts against the wall greater than for fresh earth recently deposited. Hence it will be assumed in all that follows that the filling is well drained, which can be effected by placing nearly horizontal drains near the bottom, to the front and rear of the wall, and by constructing weep holes through the wall at intervals of about 15 to 25 feet, dependent on the height of the wall and the porosity of the filling, which (next to the wall at least) should be of sand or gravel, when practicable. Such complete drainage, with a good foundation, should ensure the stability of the wall, if it has been properly designed for the thrust of the fresh earth, supposed endowed with friction but devoid of cohesion.

A sharp distinction must here be drawn between the estimation of the earth thrust against a wall for purposes of design and for the completed structure, where the earth has had time to consolidate more or less. For purposes of design, the cohesion will be generally ignored; but to estimate the actual thrust, some months or years after the wall is finished, it would be absurd to ignore the cohesion, which is, in fact, one of the most vital elements in the computation.

Another case where cohesion must be included, to interpret results properly, is that of experiments on the usual very small retaining boards backed by sand.* Experiment has shown that clean, dry sand has a coefficient of cohesion of 1 lb. per sq.

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* See an exhaustive discussion of this subject in a paper by the writer in *Trans. Am. Soc. C. E.*, vol. LXXII, entitled, "EXPERIMENTS ON RETAINING WALLS AND PRESSURES ON TUNNELS."

ft., or more, and its inclusion in the computation of the earth thrust against the model retaining boards of Leygue, the height of the sand resting against the wall being only 0.656 ft., very materially decreased the thrust and modified the interpretation of the results. For walls of 6 to 10 ft. or higher, it was found that, when backed with clean, dry sand, the cohesion could be ignored in the computation without appreciably affecting the results; hence it was advised that future experiments should be made only on walls at least 6 ft. high and preferably higher. From what precedes, it is evident that the treatment of earth pressures would be very incomplete if cohesion was ignored in all cases; hence a complete chapter will be devoted to coherent earth, with resulting applications, the remaining chapters pertaining to earth thrusts to be used in the design of walls, the earth being supposed as devoid of cohesion.

2. The Laws of Friction and Cohesion. Coulomb (1781) was the first to formulate the laws of friction and cohesion affecting a mass of earth. For homogeneous earth they may be stated as follows:

Law 1. The maximum frictional resistance along any portion of a plane in the interior of a mass of earth equals the normal pressure on the portion of the plane considered, multiplied by f, the coefficient of friction, where f is a constant for the earth considered. The friction is thus independent of the area pressed.

Law 2. The maximum cohesion equals the area considered under compression, multiplied by a constant c, the coefficient of cohesion or cohesion per unit of area. The cohesion is thus independent of the normal pressure.

It is a help, in the applications, to note that both the frictional and cohesive resistances act in the plane considered, in the opposite direction to the impending motion.

Where the cohesion is so small as to be negligible, law 1 may be said to be verified by experiment. An analogous law is given in all books on mechanics for the maximum friction that can be exerted between any two solid bodies at rest. But very few experiments have been performed to prove that law 2 is in accord with the facts or that both friction and cohesion, according to laws 1 and 2, are exerted at the same time, on the plane considered; so that the question is still an open one. A great deal of very nice experimenting will have to be undertaken to settle it finally.

It is probable, in connection with law 2, for very compressible substances, as fresh earth, that the coefficient of cohesion will increase with the normal pressure; for the area of actual contact of the particles increases with the pressure, since such pressure can squeeze the particles together and cause a more intimate contact. Therefore since the cohesion varies directly as the area of contact, it should be greater, the greater the normal pressure. As an illustration, consider fresh pulverized earth, whose coefficient of cohesion is small, though it increases slowly with time, under the influence of rains, loads, etc.; but take this fresh earth, wet it thoroughly and ram it, and it makes an excellent puddle wall, or a cut-off wall, to keep out water under very considerable pressure. Here the cohesion has enormously increased because the contact of the particles is now so much more intimate than when the earth was in a pulverulent state.

As a corollary, it looks reasonable to suppose that the cohesion in a bank of earth increases somewhat with its depth. If so, an average value will have to be taken of the coefficient of cohesion for the depth supposed, for the imaginary homogeneous earth, to which the theory pertaining to coherent earth, as given in Chapter V, alone applies. It seems highly desirable then, in experiments, to subject the earth to pressures corresponding to what it actually sustains in banks, say from 0 to 50 feet in height, to ascertain the variation, if any, in the coefficient of cohesion.

3. A simple way to find experimentally the coefficients f and c can be illustrated by Fig. 1, where a thin slice of earth is supposed to be placed between two rough metallic plaques, which are then to be firmly pressed together without contact and the resistance to the relative displacement of the two plaques

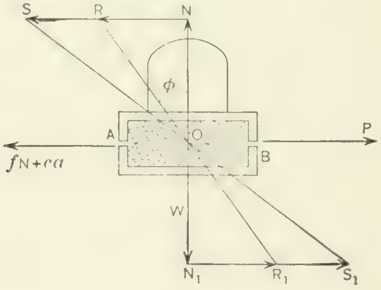

FIG. 1

recorded. Suppose the pressure to be due to a weight and let W = weight of earth and plaque above the plane of shear AB + any load resting on the upper plaque; also let P be such a force, applied to the upper plaque in the plane AB, that motion or sliding of the earth above the plane AB over the earth below it is impending. This is taken to mean, throughout the book, that no motion ensues, but that the slightest increase in P will cause motion. The earth along the plane AB resists the force P by its friction and cohesion, acting to the left along AB. The normal reaction N of the earth below AB is equal to W; hence by law 1 the frictional resistance $= fN = fW$. If now, the area of the plane AB on which shearing is impending is called a, the cohesive resistance, by law 2 is ca.

Hence for equilibrium,

$$P = fN + ca = fW + ca \quad . \quad . \quad . \quad . \quad (1)$$

When several experimental determinations of P for different values of W have been made, we can form a series of equations, of the form (1), and by elimination deduce the corresponding values of f and c.

In Fig. 1, if we lay off vertically downward, $ON_1 = W$, and horizontally to the right, $N_1S_1 = P$, then OS_1 = resultant of P and W. Also if we lay off vertically upward $ON = N = W$ and from N, horizontally to the left, $NR = fN = fW$ and $RS = ca$, then by eq. (1), $NS = fN + ca = P$ and OS the resultant of N and $(fN + ca)$, is equal and directly opposed to OS_1, the resultant of P and W. If P is $< fW + ca$, then only so much of the friction and cohesion as is needed to equilibrate P is exerted.

4. Earth Devoid of Cohesion. For non-coherent earth, $c = 0$ and $RS = 0$, whence for equilibrium $P = N_1R_1$ and OR_1, the resultant of P and W, is equal and directly opposed to OR, the resultant of N and fN.

The angle $NOR = \varphi$ is called the *angle of friction*. We have,

$$\tan \varphi = \frac{NR}{ON} = \frac{fN}{N} = f \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence for the impending motion, the resultant reaction OR always makes the angle $\varphi = \tan^{-1} f$, with the normal reaction of the plane AB. Also,

$$P = N_1 R_1 = NR = f N = f W = W \tan \varphi.$$

The same principles and formulas apply, when one solid body is on the point of sliding over another, the surfaces of contact being plane.

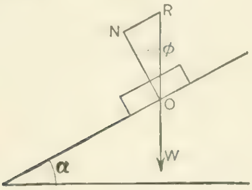

FIG. 2

5. If a block is placed on an inclined plane, Fig. 2, making an angle α with the horizontal, and slipping downward is impending, the total reaction of the plane OR must be equal and opposed to the weight W of the block which acts vertically downward. Hence, when cohesion is ignored, since OR makes the angle $\varphi = NOR$, with the normal to the plane, it follows that $\varphi = \alpha$, the sides of the two angles being perpendicular, each to each. The angle α is called the *angle of repose* for the two surfaces in contact. Since $\alpha = \varphi$, it is easy to find the coefficient of friction between any two bodies by placing one on the other and tilting the second one until motion just ensues. On measuring the corresponding angle α, we have $f = \tan \varphi = \tan \alpha$.

If the two bodies are both earth, the plane inclined at the angle φ (the angle of friction of earth on earth) to the horizontal is the plane of steepest slope on which the earth particles will not slide, and the angle φ is sometimes called the angle of natural slope.

As a corollary to what precedes, the following theorem for non-coherent earth, due to Rankine, may be stated:

THEOREM. *It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not at any point make with the normal to that plane an angle exceeding the angle of repose.*

6. The coefficients of friction for a great number of bodies have been found experimentally, generally after the method first given. Thus, as in Fig. 1, the force P is found that will just start one body sliding on another. If the weight of the upper body, with any load that may be placed on it, is W, then, $f =$

$\tan \varphi = P/W$. This ignores any cohesion. If there *is* cohesion, then this method of procedure gives too large a value for $f = \tan \varphi$, in that it substitutes NS for NR in Fig. 1, or it assumes $\varphi = NOS$ in place of the true value NOR. Thus the coefficient of "*static friction*" is always too large. When motion once begins, then presumably all cohesion is destroyed, and it would seem that the so-called coefficient of *kinetic friction* is nearer the true value of f.

7. Coefficients of Friction and Cohesion. Very few experiments have been made with the object of determining the coefficients of cohesion.

In the experiments of Leygue,* an inclined plane was used, but the laws of Coulomb, as symbolized by eq. (1), were assumed as the basis of the computation. In these experiments, the normal pressures were very small—only 7 to 30 lb./ft.^{2}—but the laws of Coulomb were verified within these narrow limits. The coefficients of friction f, with the corresponding angle of friction φ, together with the coefficient of cohesion c in lb./ft.,^{2} as computed, are as follows:

Dry sand,	$f =$ 0.70,	$\varphi =$ 35°,	$c =$ 1.47
Wet sand,	0.85,	40° 22′,	8.28
Very wet sand,	1.70,	59° 30′,	6.36
Damp fresh earth,	1.63,	58° 28′,	18.45

Leygue states that Collin found, by an independent method, for clayey earth, $c = 23.1$, and for clay of little consistency, $c = 39.5$ lb./ft.^{2}

The next series of experiments were made in 1910 by Mm. Jacquinot and Frontard, on earth taken from an earthen reservoir dam that had failed from a considerable lowering of the water level.† To the earth, water was added, and it was then worked or kneaded and compressed with the hands so as to make a firm, though pasty cake, which was then inserted between the plaques, Fig. 1. The pressures were high but the laws of Coulomb were again approximately verified. The results are as follows:

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* Annales des Ponts et Chaussées, Nov., 1885.

† Given in some detail in Résal, "Poussée des Terres," Deuxième Partie, p. 327.

Normal pressure N lb./ft.²	Coeff. friction. $f = \tan \varphi$ $\varphi = 8°$	Coeff. cohesion. c lb./ft.²
692	$f = 0.14$	395
2980	"	420
5665	"	403
7154	"	448

The earth weighed 112 lb./ft.³, as taken from the bank. The pressures recorded would correspond to the vertical pressures experienced at depths of 6 to 64 feet in a bank of the earth considered.*

The large values of c were to be expected, since the theory of open cuts or trenches would lead us to expect for ordinary consolidated earth values of c running into several hundred pounds; but the very small value of $f = \tan \varphi$ is startling, for it means that we may have to revise our ideas very greatly with reference to this coefficient of friction for even ordinary earth.

It may be noted, in this connection, that certain earths can exert an appreciable amount of tension. Thus Mr. W. Airy† states that he found the tensile strength of a block of ordinary brick clay to be 168 and of a certain shaley clay 800 lb./ft.². Since the maximum intensity of shear in a prismatic block is one-half the unit tension, it is seen, for these two specimens, that the unit shearing or cohesive strength must have been, at least, 84 and 400 lb./ft.² respectively.

Experiments have also been made by Mr. Arthur Langtry Bell, M. Inst. C. E., on certain clays encountered in sinking a monolith at Rosyth, to determine the coefficients of friction and cohesion.* The following average values of c and φ are given by Mr. Bell:

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* There were other experiments performed upon the earth as it was taken from the dam, also after it had been rammed, but the manometers were out of order and the results uncertain. It was thought, though, that the following conclusions could be fairly drawn:

(1) $f = \tan \varphi$ varied only from 0.14 for the soft, pasty earth to 0.18 for earth nearly dry. Neither amount of water used nor the puddling or ramming caused much variation in f.

(2) On the contrary, the amount of water used affects the cohesion very much, and sufficient ramming can more than double the coefficient of cohesion. If is evident that a great deal of experimental work along these lines should be done to reach safe and practical conclusions.

† Minutes of Proceedings, Inst. C. E., vol. LXV, p. 140.

	c Tons per Sq. Ft.	φ
Very soft puddle clay	0.2	0°
Soft puddle clay	0.3	3°
Moderately firm clay	0.5	5°
Stiff clay	0.7	7°
Very stiff boulder clay	1.6	16°

By "puddle clay" is meant a pure, homogeneous, plastic clay, free from sand and stones. The author states that for "perfectly dry sand," $c = 0$ and φ = angle of repose, if the sand is rammed as it is inserted in the cylinder; but if merely poured in and shaken, the experimental value of φ was much less than the angle of repose. For wet sand grabbed from a monolith well, the values found were: $c = 0.3$ ton per sq. ft., $\varphi = 31°$.

8. Tables of Angles of Repose, Weights, etc. The following table gives, for loose earth or other granular material, the *angle of repose*, or the steepest inclination of the surface to the horizon, and the weights in pounds per cubic foot.

TABLE I

ANGLES OF REPOSE AND WEIGHTS PER CUBIC FOOT

Material	Angle of Repose	Weight Lb. per Cu. Ft.
Sand, dry	20° to 35°	90 to 110
Sand, moist	30 to 45	100 to 110
Sand, wet	20 to 40	110 to 120
Ordinary earth,† dry	20 to 45	80 to 100
Ordinary earth, moist	25 to 45	80 to 100
Ordinary earth, wet	25 to 30	100 to 120
Gravel, round to angular	30 to 48	100 to 135
Gravel, sand and clay	20 to 37	100 to 115
Cinders, bituminous	25 to 41	45
Coke	30 to 45	23 to 32
Coal, anthracite	27	40 to 55
Coal, bituminous	35	44 to 54
Soft, flowing mud	0	105 to 120
Soft, rotten rock	37	110
Hard, rotten rock	45	100
Ashes, anthracite	45	30
Rip-rap	45	110

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* Min. Proc. Inst. C. E., vol. CXCIX, session 1914–15. Part *i*.

† A loose mixture of sand or gravel with clay and perhaps humus—*loam*. With 20% clay, it may be called a sandy loam; with 50% clay, a clay loam.

The foregoing table, as well as the tables that follow, have been compiled mainly from "American Civil Engineers' Pocket Book," "Trautwine's Engineers' Pocket Book," and Rankine's Treatises.

TABLE II

Coefficients and Angles of Friction

	f	φ
Masonry upon masonry	0.65	33°
Masonry upon wood, with grain	0.60	31°
Masonry upon wood, across grain	0.50	26° 40′
Masonry on dry clay	0.50	26° 40′
Masonry on wet clay	0.33	18° 20′
Masonry on sand	0.40	21° 50′
Masonry on gravel	0.60	31°
Dry bituminous coal on iron	0.84	40°
Dry bituminous coal on wood	1.00	45°
Dry anthracite coal on iron	0.58	30°
Dry anthracite coal on wood	0.70	35°

9. Weight of Granular Material in Water. Suppose the filling behind a retaining wall to consist of approximate spheres, like marbles, permitting free access to the water covering it, on all sides. If this filling weighs 100 lbs. per cu. ft. in air and has 40 per cent voids, then 1 cu. ft. of the filling contains

TABLE III

Weights per Cubic Foot of Masonry

Kind of Masonry	Weight Lb. per Cu. Ft.
Cinder concrete	110
Stone concrete	140 to 150
Reinforced stone concrete	150
Brick masonry, soft	100
Brick masonry, common	125
Brick masonry, pressed	140
Dry rubble, sandstone	110
Dry rubble, limestone	125
Dry rubble, granite	130
Mortar rubble, sandstone	130
Mortar rubble, limestone	150
Mortar rubble, granite	155
Ashlar, sandstone	140
Ashlar, limestone	160
Ashlar, granite	165

0.6 cu. ft. of solids, and since the buoyant force of the water equals the weight of an equal volume of water or $0.6 \times 62.5 = 37.5$ lbs., the true weight of the filling in fresh water is $100 - 37.5 = 62.5$ lbs. per cu. ft.*

TABLE IV

SAFE-BEARING CAPACITY OF SOILS

Soil	Tons (of 2,000 Lbs.) per Sq. Ft.
Quicksand, alluvial soils	0.5 to 1
Clay, soft	0.5 to 2
Clay and sand	2 to 4
Sand, clean, dry	2 to 4
Sand, compact, well cemented	4 to 6
Gravel and coarse sand	6 to 8
Ditto, well cemented	8 to 10
Clay, hard, moderately dry	4 to 6
Clay, hard, dry	6 to 8
Rock, soft to hard	5 to 200

This method can be applied to get the approximate weights in water of such materials as rip-rap, cobble-stones, or clean gravel. It cannot apply to the finer materials, as clean sand, bank sand, or loam, because in such earths free circulation of the water is not realized; the water cannot get under the material everywhere and its full buoyant effect cannot be exerted. However, from lack of sufficient experimental data, the method is applied to all earths, so that if the filling weighs in air w lbs. per cu. ft. and 1 cu. ft. contains n cu. ft. of solids, then, reasoning as above, the weight per cu. ft. of the filling in fresh water is,

$$(w - n \times 62.5) \text{ lbs.}$$

If p = per cent of voids, $n = (1 - p/100)$.

With this weight of filling in water, the earth thrust is computed, for the angle of repose of the earth in water and the water pressure against the wall will be estimated for the full head of water. For salt water, replace 62.5 in formula above by 64.

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* It is perhaps well to caution here against the very common error of subtracting 62.5 from the weight per cu. ft. of the filling in air to get its weight per cu. ft. in fresh water.

An interesting experiment on the transmission of pressure through saturated sand has been described by Mr. J. C. Meem,* in which it was shown that the water pressure on a given area, *through* 6 or 8 inches of sand, with 40% voids, was about 40% of the static head. The pressure averaged 4.5 lbs. per sq. in.

It will prove instructive to apply the results to the case of sand weighing 100 lbs. per cu. ft. in air, with 40% voids. The earth, in water, is assumed to transmit only 40% of the water pressure due to the full static head; consequently a cubic foot of the earth, containing only 0.6 cu. ft. of solids, will be subjected to an effective buoyant force of only $0.4 \times 0.6 \times 62.5 = 15$ lbs.; whence the earth will weigh $100 - 15 = 85$ lbs., in water. The earth thrust against the retaining wall is now to be computed for this weight per cu. ft. and the corresponding angle of repose of the earth in water. The water pressure on the wall from the surface of the saturated earth downward will then be taken as only 40% of that due to the static head.

It is to be hoped that experiments on a large scale, with varying pressures and thicknesses of sand and earth, will be undertaken to establish more definitely the laws of the transmission of water pressure through earth. In this connection, see "The Action of Water Under Dams," by J. B. T. Colman, in *Proc. Am. Soc. C. E.*, Aug., 1915.

By kind permission of Prof. Mansfield Merriman, the following table is quoted from the "American Civil Engineers' Pocket Book."

Material excavated by a wet or dry process and dumped into water, as at the back of a sea wall, has weights and slopes approximated as follows:

TABLE V

Kind of Material	Slope of Repose	Angle of Repose	Weight Lb. per Cu. Ft.
Sand, clean	2 to 1	26° 34′	60
Sand and clay	3 to 1	18 26	65
Clay	3.5 to 1	15 57	80
Gravel, clean	2 to 1	26 34	60
Gravel and clay	3 to 1	18 26	65
Gravel, sand and clay	3 to 1	18 26	65
Soil	3.5 to 1	15 57	70
Soft, rotten rock	1 to 1	45 00	65
Hard rock, rip-rap	1 to 1	45 00	65
River mud	∞ to 1	0 00	90

The slopes given were observed, the weights computed for assumed voids. In any specific case, it is desirable for the engineer to obtain the slope and weight of the earth by observation. Lacking that, it is common practice to take the weight of ordinary dry earth at 100 lb. per cu. ft. and the angles of repose as 30° or 33° 41′, the latter corresponding to the slope of 1½ base to 1 rise.

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* *Trans. Am. Soc. C. E.*, vol. LXX, pp. 366–369.

10. In the following chapters methods will be given for estimating the earth thrust for homogeneous earth against a retaining wall, but it may be well to caution the engineer here that exceptional cases arise where the ordinary theory is inadequate and which will require the exercise of his best judgment.

Thus, certain clays swell when exposed to the air with great force, or perhaps the filling may be deposited upon rock or clay sloping toward the wall and water from springs or rains lubricates the rock or clay base, lessening the frictional resistance and greatly increasing the thrust. As a simple illustration, let us suppose that the rock base makes an angle i with the horizontal, see Fig. 3, and that in consequence of springs at C, there is a break in the earth along CD and that water freely finds its way along the rock, destroying any possible cohesion there and reducing the coefficient of friction of earth on rock to $f = tan\ \varphi$.

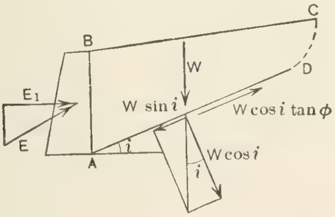

FIG. 3

Call the weight of the earth ABC that tends to slide on the rock base AD, for one foot length of wall, W, the reaction of the wall AB, E, and its horizontal component E_1. Then $W\ sin\ i =$ component of W in direction DA, and the normal component is $W\ cos\ i$; whence the frictional resistance acting up the plane or in the direction AD is $W\ cos\ i\ tan\ \varphi$, by Coulomb's law. The force tending to cause the mass $BADC$ to slide down AD is thus,

$$W \sin i - W \cos i \tan \varphi$$

and for equilibrium of this mass, its horizontal component must equal E_1.

$$\therefore E_1 = W (\sin i - \cos i \tan \varphi) \cos i = \frac{W \sin (i - \varphi)}{\cos \varphi} \cos i.$$

The direction of E will receive attention in the next article. The value found for its horizontal component is independent of that direction.

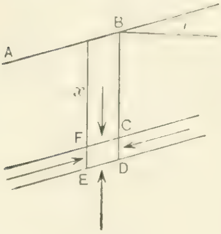

FIG. 4

11. Direction of Pressure. Consider an unlimited mass of earth with a plane upper surface AB, Fig. 4, making the angle i with the horizontal. The plane of the paper in Fig. 4 represents a vertical section cutting the surface in the line AB of greatest slope and BCD a vertical plane perpendicular to the plane of the paper. Consider the conditions of equilibrium of the prism $CDEF$, bounded by planes parallel to the surface and laterally by vertical planes, two of them, CD and EF, being perpendicular to the section, and two parallel to it, and one unit apart. Let $BC = x, CD = \Delta x$ and regard the earth as homogeneous and weighing w lb. per cu. ft.; also let the common area of the upper and lower surfaces of the elementary prism be unity. Thus if $FC = 1$ ft., the area of the horizontal section of the vertical prism BF is $\cos i$ sq. ft. and the weight of this prism is $wx \cos i$ pounds. *If the earth is exposed to no external force but its own weight*, the only pressure that the elementary prism $CDEF$ can sustain on its upper surface is the weight of earth directly over it, or $w\, x \cos i$ lbs., acting vertically downward. Similarly the downward pressure on the lower surface ED is $(x + \Delta x)\, w \cos i$ and this is equal and directly opposed to the reaction on ED, acting vertically upward. The vertical forces acting on the prism $CDEF$ (the weights of BF and EC and the reaction of ED) thus balance independently. Hence the pressures on the vertical planes CD, EF, must balance independently; consequently *they must act parallel to the sloping surface and be of equal intensity at the same depth.* The pressures acting on the vertical faces of the prism $CDEF$, parallel to the plane of the paper, are evidently horizontal, from considerations of symmetry, and balance independently.

It has now been shown that the pressure on a plane parallel to the upper plane surface of the earth is vertical and proportional to the depth; the pressure on the vertical plane CD is parallel to the upper surface and finally, the state of stress at a given depth, is uniform.

These are the conclusions of Rankine, as given in his "Applied Mechanics," Arts. 101, 125, 195, and they are sound for the case supposed, where the earth is subjected to no external force but its own weight.

The vertical pressure on CF and the pressure on CD are said to be *conjugate pressures*, each acting parallel to the plane on which the other acts. Also each stress makes the same angle i with the normal to the plane on which it acts, or they have the same *obliquity*, the angle which any stress makes with the normal to the plane on which it acts being called its obliquity.

The conclusions above hold if an incompressible earth is deposited, to the same depth everywhere, on an incompressible plane foundation, inclined or horizontal. They likewise hold for a compressible earth, even when the foundation is compressible, provided the compression of the foundation is everywhere uniform, and any plane in the earth parallel to the free surface remains plane and parallel to the surface after the compression of the earth, since the conditions assumed do not affect the reasoning.

But if the foundation gives more in some places than others, then the earth will sink more where the foundation is most yielding. This sinking is resisted to a certain extent by the friction resulting from the thrust of the earth surrounding the falling mass, so that part of its weight is transmitted to the sides, as actually happens in the case of fresh earth deposited over drains or culverts. When a tunnel is driven through earth, most of the weight of the material vertically over the tunnel is transmitted to the sides, only a small portion being sustained by the tunnel lining. Even when the foundation is practically unyielding, if it is not plane, so that the depth of earth varies very materially, the earth of greatest depth will sink most, so that part of its weight will be transmitted to the sides through friction. Thus in large masses of earth, as in embankments as we usually find them, the pressures are not generally the same at the same depth. In fact, the conditions originally posited are rarely fulfilled in ordinary practice.

12. **Direction of the Thrust Against a Retaining Wall.** If we imagine an immovable rigid wall on an unyielding foundation, backed by an incompressible granular mass having a plane free surface, then the pressure of the filling on a vertical plane entirely in the earth and touching the wall will be parallel to the free surface, by the preceding theory. But in practice, the wall, filling, and foundation are compressible, and almost invariably the filling is much more compressible than the wall, so that it settles more than the wall. As a consequence, the earth rubs against the wall in its relative downward motion, developing friction and thus introducing an extraneous force not contemplated in the foregoing (Rankine) theory. Again, the wall and foundation being compressible, the top of the wall moves over slightly from the earth thrust, if the resultant of this thrust and the weight of the wall passes outside the center of the base of the wall. It is only for certain batters of the back of the wall (to be investigated in Arts. 27–30), that a certain wedge of earth moves with the wall as it moves over, in which case the earth sooner breaks along an interior plane in its mass than along the wall. Excluding these cases for the present, for all other cases, as the top of the wall moves over, however slightly, or strictly, when such motion is impending, the earth will get a grip on the wall, acting downward along the back of the wall, equal to the true normal component of the earth thrust multiplied by the coefficient of friction of earth on masonry, and it is impossible for the wall to overturn or slide without this full frictional resistance being exerted.

If the top of the wall leans toward the earth, this theory is only approximately true. If the back of the leaning wall makes an angle with the vertical not exceeding about 10°, or for usual inclinations, the theory is sufficiently near for computation of the resulting thrust. See Art. 62. Let us apply this theory to the retaining wall $ABCD$, Fig. 5, backed by earth ABL, having an angle of friction φ, the angle of friction of earth on wall being φ', all cohesion being neglected. If the true normal component of the earth thrust on AB is called E_1, then from the tendency to relative settling of the earth or from the impending moving over of the top of the wall or from both causes, the earth will exert a friction force, acting always opposed to the incipient motion, or along BA, in the direction BA, of amount E_1 *tan* ϕ' or E_1 *tan* φ, if $\varphi' > \varphi$. In the latter case ($\varphi' > \varphi$), a thin layer of earth will cling to the wall, so that in case of the impending relative motion supposed, this layer, rubbing against the remaining earth, will only cause the friction due to earth on earth. Let us suppose $\varphi' \leqq \varphi$, then the friction force is E_1 *tan* φ'. The reaction MI of the wall can be decomposed into two components; one normal to the wall, $NI = E_1$, since it is equal and opposed to the normal component of the earth thrust; the other OI, equal and opposed to the friction force E_1 *tan* φ'.

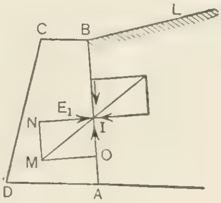

FIG. 5

Completing the parallelogram $MOIN$, we have $MN = OI = E_1 \tan \varphi'$; whence,

$$\tan NIM = \frac{MN}{NI} = \frac{E_1 \tan \varphi'}{E_1} = \tan \varphi'$$

$$\therefore NIM = \varphi',$$

or, the reaction of the wall makes the angle φ' with the normal to the wall. When $\varphi' > \varphi$, as shown above, the friction force is E_1 *tan* φ; hence in this case $NIM = \varphi$.

It is thus seen that the reaction of the wall, when friction at the back of the wall is exerted, makes the angle φ' with the normal to the back of the wall when $\varphi' \leq \varphi$, or the angle φ with the normal when $\varphi' > \varphi$. Consequently, when AB is vertical, the total earth thrust on the wall, which is equal and directly opposed to MI, the reaction of the wall is not parallel to the free surface, as in the Rankine theory, save when the free surface is at the angle of repose and $NIM = \varphi$. The extraneous force—the wall friction—causes this change in the usual direction of the earth thrust on a vertical plane.

It will be observed, in the analysis above, that no actual relative motion is required, but only "impending" relative motion of earth and wall.

13. It will now naturally be inquired, What has experiment to say as to the direction of the earth thrust against a vertical retaining wall? Is it parallel to the free surface or does it make an angle φ' or φ ($\varphi' > \varphi$) with the normal to the wall? In answer to this question in a general way, let us consider a very light, empty wooden box of a width slightly greater than the height and of any length and backed by sand level with the top of the box. If the box is kept from sliding by a small obstacle placed at the exterior edge, which does not interfere with rotation, the box will not overturn from the thrust of the sand.

By the Rankine theory, the earth thrust against the vertical side of the box is parallel to the free surface or horizontal. This, combined with the weight of the box, which is practically negligible, gives a resultant nearly parallel to the base, so that overturning should occur. Since the wall is stable, the evidence is conclusive that the assumed direction of the thrust is not the true one. But if we assume the friction coefficient of sand on wall as about $\frac{1}{3}$, then the resultant of the earth thrust, inclined below the horizontal, at the angle whose tangent is $\frac{1}{3}$, with the weight of wall, acting along the vertical through its center of gravity, will pass inside the base of the wall and stability is assured. The wall-friction method is thus not inconsistent with the facts if the friction coefficient is about $\frac{1}{3}$. It was not given by Flamant, who recorded the observation.

The author has investigated the results of experiments pertaining to the case of retaining walls or boards at the limit of stability and has recorded his conclusions in a paper entitled "Experiments on Retaining Walls and Pressures on Tunnels,"* to which the reader is referred for quantitative results. The conclusion is drawn from the experiments cited on retaining *walls* that the direction of the thrust as given by Rankine is never experienced in walls at the limit of stability, except where the surface slopes at the angle of repose. When, however, the wall-friction is included, so that the thrust is supposed to make the angle φ' (when $\varphi' \leqq \varphi$) or φ (when $\varphi' > \varphi$) with the normal to the wall, the results agree fairly well with the experiments, the backing in every case being clean sand. In the case of the rotating *boards*, the models were so small that cohesion appreciably affected the results. But with an assumed cohesion of 1 lb. per sq. ft., it was found that the direction of the thrust was approximately as given by the last method.†

14. Now, if it is granted that the wall-friction method best explains the results of experiments on walls *at the limit of stability*, it is still doubted by some whether this friction can always be relied on in the case of a stable wall throughout the years of its existence. Let us consider the somewhat analogous case of the narrow railroad embankment over which pass heavy locomotives at great speed. The very integrity of this embankment is dependent on the permanence of the friction coefficient, and it never fails when the foundation is good and it is well drained, in spite of the vibration to which it is subjected. Or again, take the wheat in a tall bin. Hundreds of experiments have shown that most of the weight of the wheat is carried by friction by the sides of the bin even when the wheat is running out. Again, the friction factor is permanent and the bin would be quickly destroyed if it was not so.

But, it may be argued, may not very heavy rains, which the drains provided may not carry off quickly enough, decrease φ and φ', and thus not only lead to an increase of the thrust but likewise cause it to make a less angle with the horizontal than before?

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* *Trans. Am. Soc. C. E.*, vol. LXXII, p. 403.

† See Appendix II for a discussion of a few of the experiments.

Let us consider the two possibilities separately.

From a consideration of the angles of repose in Tables I and V, it is seen, for an average clean sand, that the values of φ, for the states dry, moist, wet and submerged, will be something like the following:

$$33^\circ\ 41', \quad 44^\circ, \quad 33^\circ\ 41', \quad 26^\circ\ 34',$$

and for ordinary earth, say,

$$37^\circ, \quad 39^\circ, \quad 30^\circ, \quad 17^\circ.$$

For hard rock or rip-rap, there is but little change. For gravel or mixtures of sand, clay, and gravel, the changes are not so pronounced as in the case of ordinary earth.

From these results, it is seen, for well-drained filling, that for average materials there is an appreciable decrease in φ as the material changes from dry to wet. This could be met by designing the wall for the wet filling, in the first instance. Dock walls should of course be designed for the values of φ corresponding to filling under water, for the part of the filling that is submerged and for wet filling for the part above that. But where it is only in times of excessive floods that an ordinary retaining wall is submerged or partially submerged, it seems unnecessary to design such a wall as one would a dock wall. The failures from such excessive floods are too few to warrant the extra cost; in fact, it will be more economical to use loose rock filling, in part at least, for the value of φ for such filling is about the same, dry, wet, or submerged.

As to the second possibility, it seems reasonable to suppose that heavy rains may lubricate the wall to such an extent that φ' is materially decreased. On that account and to allow, too, for vibration, due to heavy passing loads, it is recommended to multiply only the normal component of the thrust E_1 by the factor of safety σ. Then on combining $\sigma\ E_1$ with the friction component $E_1\ tan\ \varphi'$ (or $E_1\ tan\ \varphi$, when $\varphi' > \varphi$), and this resultant in turn, with the weight of the wall, acting along the vertical through its center of gravity, let the wall be given such a thickness, that this final resultant shall pass exactly through the outer toe. So far as the direction of the thrust is concerned, this method is equivalent to replacing *tan* φ' by *tan* φ'/σ. Thus if $\sigma = 3$, for walls over 10 ft. high, only one-third of the full friction is included in the moment formula. This seems ample in allowing for heavy rains and perhaps vibration.

We know very little as to the influence of vibration on earth thrust. It may cause minute movements of the wall from time to time, from which it does not recover. In fact, if the filling follows up such movements, it is difficult to see how the wall can regain its original position.

Trautwine* says, with reference to his little wooden retaining walls, about 6 inches in height and backed by sand: "We found that the tremors produced by passing vehicles in the street, by the shutting of doors and walking about the room, sufficed to gradually produce leaning in walls of considerably more than twice the mere balancing stability while quiet." The models were in the upper room of a strongly built house.

This observed increased leaning with time of actual retaining walls may be largely due to the vibration caused by heavy passing loads and certainly the factor of safety for walls 6 to 10 ft. high should be larger than for walls over 10 feet; say, at least 3.5.

It is a common and commendable practice to step the back of a retaining wall, as illustrated in Fig. 6. Any moving over of the top of the wall will tend to cause a break in the earth along a plane touching the back of the wall, since the triangular portions of earth over the steps will move with the wall. The friction developed is thus of earth on earth, so that φ' is replaced by φ. If φ' for a smooth concrete wall, lubricated with water in times of heavy rains, is much smaller than φ for the wet earth, there is decided economy in stepping the wall, since the direction of the thrust on the back face, is more inclined to the horizontal.

15. Unit Stresses on a Horizontal Section. Let $A\ D\ E\ B$, Fig. 6, represent a portion of a retaining wall contained between two vertical parallel planes, one foot apart, and perpendicular to the front and rear faces. In fact, throughout the book, one foot length of wall will always be understood, when the cross-section is uniform, or there are no buttresses or counterforts.

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* Engineers' Pocket Book, 19th ed., p. 605.

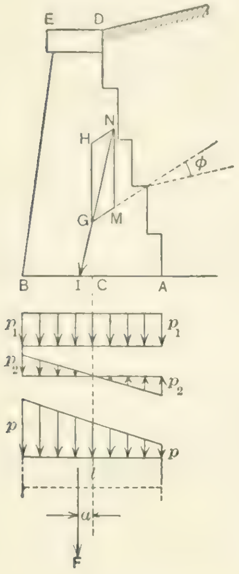

FIG. 6

By methods to be given later, the earth thrust on the back face AD can be estimated. Both the Rankine and the wall-friction methods will be given, so that the reader can take his choice—suppose the latter method to be employed; then if the rear face is plane, compute the thrust on it for the kind of earth supposed, the thrust making the angle φ' (when $\varphi' \leqq \varphi$) or φ (when $\varphi' > \varphi$) with the normal to the rear face; but if the rear face is stepped, compute the thrust on a plane, say AD, having about the average batter of the face, the thrust making the angle φ (of earth on earth), with the normal to AD.

The earth thrust, acting on the rear face, at one-third the height of wall, is produced to meet the vertical through the center of gravity of the wall at G. On laying off GH, to scale, to represent the weight of the wall and GM to represent the earth thrust, and drawing $MN \parallel GH, HN \parallel GM$, to intersection N, the diagonal NG of the parallelogram is seen to represent the resultant on the horizontal section AB. Let us suppose that it cuts this section at I, a distance a feet to the left of its center C. Also let

F = vertical component of the resultant GN on AB,
l = length of AB in feet.

If we decompose at the point I the resultant NG into a horizontal component and a vertical component F, the former is resisted by the shear along the section AB of the wall, or if AB is the bottom of the wall, it is resisted by the friction between the wall and the earth on which it rests. There remains the vertical component F to consider. At the center C of the section AB, conceive applied two vertical opposed forces, each equal to F. This does not affect equilibrium. The force F acting downward at C may be supposed to be the resultant of a uniformly distributed stress, $p_1 = F/l$, on the section, as shown by the arrows just below AB. The force F at I, acting down, with the remaining force F at C, acting up, constitutes a couple whose moment is, $F.\ IC = Fa$.

For a monolithic wall, with front and rear faces vertical, this couple will cause a uniformly varying stress (as shown by the little arrows just below the first) whose intensity at A and B is, by known laws,

$$p_2 = \frac{Fa}{l^2/6} = \frac{6\,Fa}{l^2}.$$

Evidently when I is to the left of C, p_2 is compressive at B and tensile at A. The total unit stress at B and A is thus,

$$p = p_1 \pm p_2 = \frac{F}{l}\left(1 \pm \frac{6a}{l}\right). \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

the + sign referring to the point B, the − sign to the point A. The lower arrows, contained in a trapezoid, represent the distribution of the stress, corresponding to what is called the "trapezoid law," F is the resultant in position and magnitude of these stresses. The same formula (3) is supposed to hold when AB is the plane of contact of the wall with the earth, the values of p giving the vertical components of the "soil reactions" at B and A (acting of course upward) when the entire base is in compression. It will be observed from (3), that p is positive at A when $a < l/6$ or when the resultant on AB cuts it within its "middle third."

When $a = l/6$, $p = 0$ at A and $p = 2\frac{F}{l}$ at A. Thus the unit pressure at A is double the average and the trapezoid reduces to a triangle.

When $a > l/6$, or the resultant on AB passes outside of its middle third, then, when the masonry can resist the tension, the stress at A is minus, indicating tension and is given by (3). Since the soil can only resist compression, formula (3) does not apply to finding soil reaction when $a > l/6$. For this case and also for a masonry section which cannot resist tension, the unit stresses are supposed to vary as the ordinates of a triangle. Hence, since F is their resultant, it must act through the center of gravity of the triangle and the section will sustain compression only over the length, $3.BI$, Fig. 7, and the maximum intensity at B is double the mean, or,

$$p = 2 \frac{F}{3 . BI} = \frac{2F}{3b} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

on putting $BI = b$. The stress on AB, at the distance $3\ BI =$

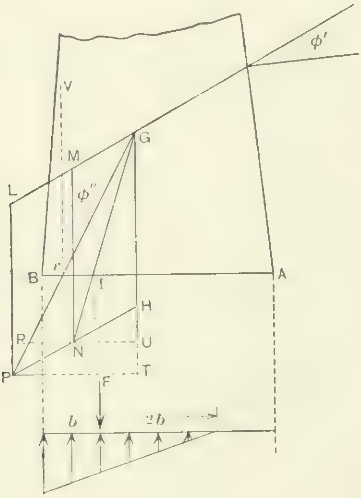

FIG. 7

$3\ b$, from B is zero, and to the right of this point there is no stress and the joint there will open.

The formulas above have been derived for rectangular walls. They are approximately true for battered walls, where, as usual, the batter does not exceed a few inches per foot, and they will be used for the ordinary types of walls throughout. The value given above for p_2 is strictly true only for rectangular walls and departs more and more from the truth as the batter increases, so that the trapezoid law is only approximately true. From the experiments on india-rubber dams, mentioned in Appendix I, it is seen that the vertical components of the stresses at the outer toes are less than the values corresponding to the trapezoid law, so that the approximation is on the side of safety. In reality, the stresses are greater in the body of a retaining wall (or dam), and less at the faces, than the theory calls for.

16. Factors of Safety Against Overturning and Sliding. *The factor of safety against overturning* is generally defined to be the number by which the earth thrust GM, Figs. 6 and 7, must be multiplied, so that the increased thrust, when combined with the weight of the wall GH, shall give a resultant that passes through B, the outer toe of the wall. See Art. 14 for another definition, which applies more particularly when the wall friction is included in computing the thrust.

To define *the factor of safety against sliding*, consider Fig. 7, where GM represents the earth thrust, GH the weight of wall, and $GU = F$ the vertical component of the resultant GN on the base AB.

Lay off the angle GrV with rV, the normal to AB, equal to φ'', the angle of friction of masonry on earth. Extend Gr to meet HN produced at P and draw PL vertical to meet GM produced at L and PT horizontal to meet GH produced at T. Then if the earth thrust was GL, the resultant on the base would be GP and sliding would be impending. Hence the factor of safety s against sliding is,

$$s = \frac{GL}{GM} = \frac{PT}{NU}.$$

If we extend UN to R to meet GP, we note that the total friction that can be exerted along the base is, $F \tan \varphi'' = GU \tan \varphi'' = UR$. The ratio of this to the horizontal component of the thrust NU is sometimes given as the factor of safety; call it t.

$$\therefore t = \frac{UR}{NU} = \frac{GU \tan \varphi''}{NU} = \frac{\tan \varphi''}{NU/GU} = \frac{\tan \varphi''}{\tan GNM}$$

We always have $t < s$, except when the thrust is horizontal, when they are equal. The last formula supposes the vertical component of the earth thrust to remain constant when the horizontal component NU is multiplied by t to give $RU = F \tan \varphi''$. The formula thus gives a slight excess, and it is often the most convenient formula to use. If practicable, the factor of safety against sliding should be 2. As this is often difficult to attain, recourse is had to projections of masonry on the base; otherwise the base is tilted up, so that GN makes a sufficiently small angle with the normal to the base, always much less than φ''.

17. Additional Remarks Relative to the Middle Third Requirement and the Use of Eq. (3). In Art. 9, Table IV, the bearing capacity of various soils is given. If, for a particular soil, the maximum pressure found from eq. (3) is greater than is permissible, then the base must be made larger. This is generally effected by building a footing or foundation course projecting beyond the front face of the wall. In this way, too, the resultant on the earthen base can be brought very near its center. To cause this resultant to pass near the center may entail an extra large footing, but if the additional expense is not prohibitive, it should positively prevent that increased leaning with time, observed in the case of some walls.

It is usually specified that the resultant on the base of a wall shall cut it within its middle third. In case of a limited right of way, it may be allowed to pass slightly outside of this limit in the case of a rock foundation, since stability is assured. But for ordinary earth, particularly if the soil resistance is uncertain, the requirement should generally be observed, and it is further advised to build the projecting footing course mentioned, so that the resultant on the base shall pass as near its center as is practicable, thus ensuring a more nearly uniform pressure on the base.

CHAPTER II

NON-COHESIVE EARTH. GRAPHICAL METHODS

18. Surface of Rupture. A large number of interesting experiments were made by Leygue* on boards backed by clean sand, having but little cohesion, to determine the surface of rupture in the sand when the retaining boards were rotated about their lower edges. The box in which the sand was confined had glass sides, the pivoted boards in front representing the retaining walls. To observe the relative movement of the grains of sand as the wall moves over, thin horizontal strata of pulverized white plaster were placed in the sand at various heights. As the retaining board moved over, it was observed that all the sand particles lying between the retaining board and a surface convex upward and passing through the foot of the board, moved parallel to this surface. This surface, called *the surface of rupture*, remained invariable as the rotation was increased up to a certain point (30° for a horizontal surface of earth); but on completing the rotation by lowering the retaining board to the horizontal, the surface moved to the rear and the earth finally took its natural slope.

The same results were found to hold when the retaining board was moved parallel to itself, except that the surface of rupture tended from the start to move to the rear.

It was further observed that the full friction of the earth was exerted against the wall in either movement.

The height of the retaining boards was varied from 0.66 to 0.82 feet for the first series of experiments, the boards being first put at the inclination desired before rotating.

In the next set of experiments the boards were 3.28 and 6.56 ft. high, with the filling level with the top and also with a surcharge up to 3.28 feet. It can be shown that the influence of cohesion is very small for the sand or broken stone used for the walls 6.56 ft. high. For the larger heights, the surface of rupture was more irregular than for the smaller heights, but it was always, taking the average, convex upward (or concave downward) not only for the sand, but also for the stones which varied from 1.5 to 20 inches in diameter. The convexity of the surface of rupture was only slight, but it was invariable.

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* Annales des Ponts et Chaussées, Nov., 1885.

For *consolidated* earth, it is well known from observation of the caving of banks that the surface of rupture is concave upward. Such earth is largely endowed with cohesion.

In this chapter, cohesion will be neglected and the surface of rupture will be taken as plane. The theory developed, for the cases represented by the experiments at least, will thus be approximate, but it is probably a close approximation for clean sand or rock filling, for walls 6 to 10 ft. high and upward. The cohesion is negligible for such heights, and the surface of rupture, for such materials, is nearly plane.

For fresh earth deposited behind a retaining wall, there is much more cohesion than for clean sand, Art. 7, so that the computed thrust, neglecting cohesion, will be somewhat in excess, particularly for heights of wall of only 6 to 10 ft., and this excess will become larger with time (except for the influence of heavy rains) since the cohesion generally increases with time.

19. Earth Endowed with Friction but Devoid of Cohesion. Sliding Wedge Theory. Let AB, in Fig. 8, represent the inner face of a retaining wall, backed by earth whose surface is BCD.

The angle of friction of the earth $= \varphi$, the angle of friction of earth on wall $= \varphi'$. Let weight of earth per cubic foot $= w$. As usual, one foot length of wall perpendicular to the plane of the paper will be considered. Suppose the earth tends to slide down a plane of rupture AC. The weight of the wedge of rupture $ABC = w \times$ area $ABC = W$. As shown in Art. 12, the reaction E of the wall (equal and opposed to the earth thrust on the wall) makes the angle φ' with the normal to AB

when $\varphi' \leqq \varphi$ or the angle φ when $\varphi' > \phi$. See Art. 30 for exceptions to this rule.

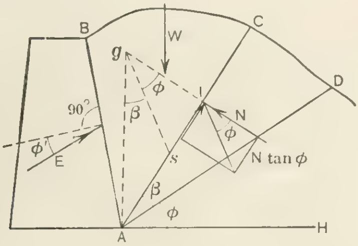

FIG. 8

In the case of experimental walls, φ' should be determined by experiment. For an actual rough masonry wall, where φ' is not known, it is usually replaced by φ. For stepped walls, φ' is always to be replaced by φ.

To allow for the lubrication of the wall with water or for excessive vibration, some designers take $\varphi' = 0$. See Arts. 12–14. For walls leaning toward the earth, the direction of the thrust can be taken approximately as above when the inclination of the wall to the vertical does not exceed about 10°. For greater inclinations see Art. 62.

The reaction of the earth on AC consists of a normal component N and a frictional component $N \tan \varphi$, acting up along AC, the resultant of the two, thus making an angle φ with the normal to AC. To find easily the direction of this resultant reaction, lay off $HAD = \varphi$ and let $DAC = \beta$. Then, if from any point g in a vertical through A, we lay off $Ags = \beta$ it will follow that gs is the direction of the resultant reaction on AC. This follows, because if gI is perpendicular to AC $\therefore$ $AgI = HAC$, the sides being perpendicular each to each. Hence $AgI = \beta + \varphi$, and if we lay off $Ags = \beta$, then $Igs = \varphi$ or gs is inclined at the angle φ with the normal to AC.

This reaction is not drawn in true position, since it must pass through the intersection of E and W, the three forces, E, W, and the reaction, being in equilibrium.

20. Active and Passive Thrust. The thrust E, Fig. 8, caused by the tendency of the mass ABC, from its weight, to slide *down* the plane AC, is called the *active* thrust of the earth against the wall. Here the friction component of the reaction of AC acts *up* or, as usual, always opposed to the impending motion.

If, however, the wall is subjected to a force acting toward the earth, which exceeds the active thrust, it will bring into play the passive resistance of the earth to sliding *up* some plane, generally below the plane of rupture for active thrust.

When the force applied to the wall is so great that motion up some plane as AC is impending, then the frictional component of the reaction of AC, $N \tan \varphi$, acting opposed to the impending motion, will act *down*, so that the resultant reaction now lies above the normal to AC and makes the angle φ with it.

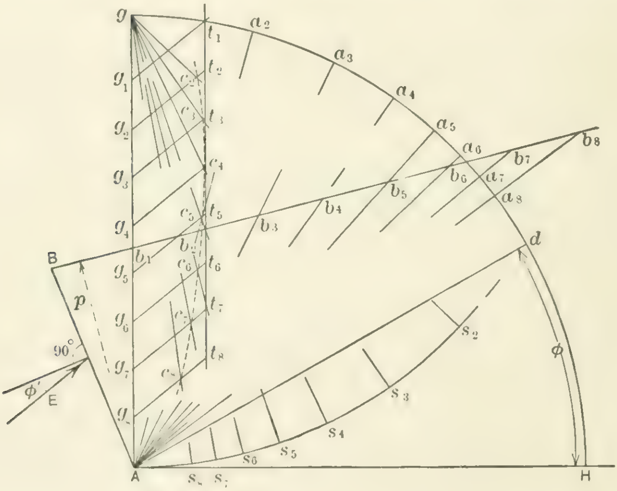

FIG. 9

The force acting on the wall is equal and directly opposed to the resistance or so-called *passive thrust* of the earth on the wall.

21. Graphical Determination of Active Earth Thrust Against a Wall. In Fig. 9, let AB represent the inner face of a wall, backed by earth whose surface Bb_8 has a uniform slope. Let us suppose $\varphi' \leqq \varphi$ and that AB lies at or above the "limiting plane" of Art. 27. Let the vertical Ag intersect the free surface at b_1 and lay off $b_1b_2 = b_2b_3$, etc., along b_1b_8.* Draw the lines Ab_2, Ab_3, etc., to represent possible planes of rupture. Let the length of the perpendicular from A upon $Bb_8 = p$; then the weight of the prism $ABb_1 = \frac{1}{2}\ w.p.\ Bb_1$, weight of Ab_1b_2 = weight of Ab_2b_3 = etc. = $\frac{1}{2}\ w.p.\ b_1b_2$. Then, by addition, we can compute the weights in pounds of the successive trial prisms of rupture ABb_2, ABb_3, etc.

On any convenient vertical as Ag, lay off gg_1 = weight of ABb_1, gg_2 = weight of ABb_2, gg_3 = weight of ABb_3, etc.

With A and g as centers and any convenient common radius describe arcs Ha_2 and As_2. Lay off the angle $HAd = \varphi$ and with dividers lay off chords As_2, As_3, As_4, etc., equal to chords da_2, da_3, da_4, etc. The points a_2, a_3, etc., are where the lines, Ab_2, Ab_3, etc., produced when necessary, cut the arc Ha_2.

Since the construction gives $\angle\ Ags_2 = \angle\ dAa_2$, etc., by Art. 19, it is seen that gs_2, gs_3, etc., make the angle φ with the normals to the planes Ab_2, Ab_3, etc., respectively. Hence gs_2, gs_3, etc., have the directions of the reactions of the successive trial planes of rupture Ab_2, Ab_3, etc.

Next draw a line inclined at the angle φ' to the normal to AB to represent the direction of E as shown and through g_1, g_2, g_3, etc., draw lines parallel to E. Let such lines through $g_2, g_3, \ldots, g_8$ intersect $gs_2, gs_3, \ldots, gs_8$, at $c_2, c_3, \ldots, c_8$.

Now observe, for any supposed plane of rupture Ab_3, that the corresponding prism of rupture ABb_3A is held in equilibrium by its weight, the reaction of the wall and the reaction of the plane Ab_3 and that these three forces are represented to scale by the sides gg_3, g_3c_3 and c_3g of the force triangle gg_3c_3. Similarly for a supposed plane of rupture Ab_4, the forces acting on the corresponding wedge of rupture ABb_4A, are given, to scale, by the sides of the triangle gg_4c_4, gg_4 representing its weight, g_4c_4 the reaction E of the wall and c_4g the reaction of the plane

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* It is usually best to lay off from B along Bb_8 equal distances of 1 or 2 feet, so that the weights of the prisms ABb_1, Ab_1b_2, etc., are all equal, and the weights of the successive prisms, ABb_1, ABb_2, etc., can be found by simple addition. In the figure, Ab_1 was taken as vertical simply for purposes of comparison with the results pertaining to Fig. 10.

Ab_4, since gc_4s_4 was drawn making the angle φ with the normal to the plane Ab_4 and is thus parallel to its reaction. Of all the trial prisms of rupture, there can be only one true one with its resulting thrust, and this corresponds to the greatest of the lines g_2c_2, g_3c_3, g_4c_4, etc., which, in this example, is g_4c_4, and thus represents the true value of E.

This follows from Rankine's principle of Art. 5, because for any less thrust than g_4c_4, the point c_4 will fall to the left of its present position, hence the new gc_4 will make an angle greater than φ with the normal to the plane Ab_4, which is inconsistent with the laws of stability of a granular mass as stated in Art. 5.

On drawing a curve through c_2, c_3, . . . , c_8, and a vertical tangent t_1t_8 to it, the greatest of the lines g_2c_2, g_3c_3, . . . , can be readily found. In any practical example, it is well to draw other trial planes of rupture near Ab_4, the one corresponding to g_4c_4, to determine by the construction more accurately the true thrust and the true plane of rupture. In fact, after making a number of constructions similar to the above, for various inclinations of AB and Bb_8, the probable position of the true plane of rupture can be guessed at rather closely, so that only a few trial planes need be drawn near this position. It is possible that the plane of rupture determined by this construction, may differ in direction 1° or 2° from the true one, but the earth thrust is determined more accurately than by any other graphical method, since intersections c_2, c_3, . . . , are determined very closely, the angles g_2c_2g, g_3c_3g, . . . , being large.* The principle demonstrated above that the true active earth thrust against the wall is the maximum of the thrusts corresponding to the trial prisms of rupture was originally given by Coulomb as a self-evident first principle and is known as Coulomb's "wedge of maximum thrust."

Having found the true reaction E of the wall (equal and opposed to the earth thrust on the wall) which equals g_4c_4 to scale and the plane of rupture Ab_4, to find the true resultant thrusts on the planes Ab_1, Ab_2, . . . , Ab_8, we extend the lines through g_1, g_2, . . . , g_8, that are parallel to E, to t_1t_8, meeting it at the points t_1, t_2, . . . , t_8. Then $g_1t_1 = g_2t_2 = g_3t_3 = g_4c_4$, etc.; hence since the resultant of gg_1 and g_1t_1 gives the resultant thrust on the plane Ab_1 and similarly for the other planes, it follows, if the straight lines t_1g, t_2g, . . . , t_8g, are drawn, they represent the actual reactions on the planes Ab_1, Ab_2, . . . , Ab_8, respectively.

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* The method used above for ascertaining the trial thrusts is analogous to that first given by Prof. H. T. Eddy in "New Constructions in Graphical Statics."

These lines (not drawn in the figure), other than c_4g, all make less angles with the horizontal than the trial values of the reactions and thus make less angles than φ with the normals to their corresponding planes. Hence the conditions of stability are all satisfied, and if the wall gives, sliding will occur only down the plane of rupture Ab_4.

To ensure accuracy in this construction, the scales of force and length should be large and the radius AH should be preferably 5 or 10 inches or more. The length g_4c_4, in this case, to the scale of force gives the thrust E in pounds—similarly the length of t_1g, to the scale of force, gives the thrust on the vertical plane Ab, in pounds. It is seen not to be parallel to Bb_8; hence it does not agree in direction with the Rankine thrust on a vertical plane, which is parallel to the top slope.

22. Variation of E with φ'. In Fig 9, if we suppose φ' to diminish, the lines g_2c_2, g_3c_3, etc., parallel to E, will approach nearer the horizontal, the points c will recede from Ag, hence the thrust, represented by the greatest of the lengths g_2c_2, g_3c_3, . . . , for a given φ', will have a greater horizontal component than before. On the contrary, if φ' increases, the points c will approach the vertical Ag and the horizontal component of the thrust will diminish.

23. Examples. In all the examples below, take $\varphi = \varphi' = 33° 41'$, corresponding to a slope of 1½ base to 1 rise.

Ex. 1. Let the inner face AB of a wall 20 ft. high have a batter of 3.75 in. to the foot or a horizontal projection of 6.25 ft. and suppose the earth, weighing $w = 100$ lb. per cu. ft., to have a free plane surface, extending from B above the horizontal and making an angle with it of 6°. Prove by the construction of Fig. 9, that the earth thrust on AB, inclined at the angle φ to its normal, is 9560 pounds, the plane of rupture meeting the surface about 20 ft. from B.

Exs. 2. In the table are given the values of E for six different batters of

AB, for $\varphi = \varphi' = 33° 41'$, $w = 1$ and the earth surface level with B, the vertical height of AB being 10 ft.

Batter of AB	E (for $w = 1$)	K	K_1
0	13.0	0.130	0.109
1 in. to 1 ft	15.1	0.151	0.126
2 in. to 1 ft	17.1	0.171	0.143
3 in. to 1 ft	19.7	0.197	0.164
4 in. to 1 ft	22.4	0.224	0.187
5 in. to 1 ft	25.2	0.252	0.210

To find the value of E when the earth weighs w lb. per cu. ft., multiply the tabular value by w to get the thrust in pounds. The columns K and K_1 are inserted for future reference, K being the thrust for $w = 1$ and height of $AB = 1$, $K_1 = K \cos \varphi$, being its component normal to AB, as will be explained in Art. 24.

Exs. 3. In the next table, the values E refer to walls 10 ft. high, whose tops *lean toward the earth* the amount given, expressed as in the case of batter. As before, the earth thrust on AB is supposed to make the angle φ with its normal; $\varphi = \varphi' = 33° 41'$, $w = 1$ and the free surface is level with B.

Lean of AB	E (for $w = 1$)	K	K_1
1 in. to 1 ft	11.4	0.114	0.095
2 in. to 1 ft	9.7	0.097	0.081

The values of K and K_1 for any intermediate batters can be interpolated from the tables with sufficient accuracy.

If the student will do careful drawing, he should be able to reach the above results on a drawing that can be contained on a sheet of writing paper. When the earth slopes downward from B, the construction of Fig. 9 equally applies for the various inclinations of AB.

The construction of Fig. 9 fails for cases where the free surface approaches the natural slope, since the plane of rupture then approaches the natural slope. In such cases we have to resort to formulas to be derived later.

24. Center of Pressure. In Fig. 9, let h = vertical height of AB, the free surface of earth being either level or having a uniform slope, ascending or descending from B. Then for the same inclination of AB, as h varies, the plane of rupture remains at the same inclination, hence the sections of the prisms of rupture for different values of h are similar triangles, so that the thrusts E, which vary directly with the weights of the corresponding prisms, will also vary as the areas of these triangles or as the squares of AB or as h^2. Hence the thrust can be represented by an expression of the form,

$$E = K w h^2$$

where K is a constant for a particular surface-slope and inclination of AB. It follows, as in the case of water pressure, that the resultant E is applied at one-third the height of the wall from the base or that *the center of pressure* is found at a point on AB, $(1/3)\ AB$ from A.

25. Definition and Use of Quantities K and K_1. Referring to the above formula, we see, when $w = 1$, $h = 1$, that $E = K$, or *K is the thrust on a wall 1 ft. high, when $w = 1$*, and that its normal component is $K \cos \varphi$, or, $K \cos \varphi'$ when $\varphi' < \varphi$. Putting K_1 for this normal component, we have,

$$K_1 = K \cos \varphi, \text{ or, } K_1 = K \cos \varphi' \ (\varphi' < \varphi).$$

It is easy to find K by the previous construction, assuming $w = 1$, $h = 1$; or we may assume $w = 1$, $h = 10$, giving the values of E in the table of the preceding article. Then $E = Kh^2 = 10^2\ K$; hence, $K = E/100$. In this way the values of K in the table were derived.

From this table, we can find the value of the thrust for any height, for $\varphi = \varphi' = 33° 41'$, earth surface level from B and a given w and inclination of AB. Thus for $w = 100$, a batter of 3.75 inches to the foot and $h = 10$ ft.,

$$K = 0.197 + \tfrac{3}{4}\,(.027) = 0.217$$

$$\therefore E = K w h^2 = 0.217 \times 100 \times 10^2 = 2170 \text{ lbs.}$$

Similarly for the same batter and value of w, but for a height of $h = 20$ ft., as in Ex. 1, Art. 23,

$$E = 0.217 \times 100 \times 20^2 = 8680 \text{ lbs.}$$

26. Thrust on a Vertical Plane in an Unlimited Mass of Earth. In Fig. 10, let $5\,Ob_5$ represent the free surface—supposed plane—and let AO represent a vertical plane in it, the earth being of unlimited extent and subjected to no external force but its own weight. Then by Art. 11, the earth thrust on AO acts parallel to the free surface or to the line of greatest declivity $5\,Ob_5$.

The construction for the earth thrust caused by the wedge of maximum thrust to the right of AO is similar to that of Fig. 9 and will be only briefly indicated. The surface Ob_5 is

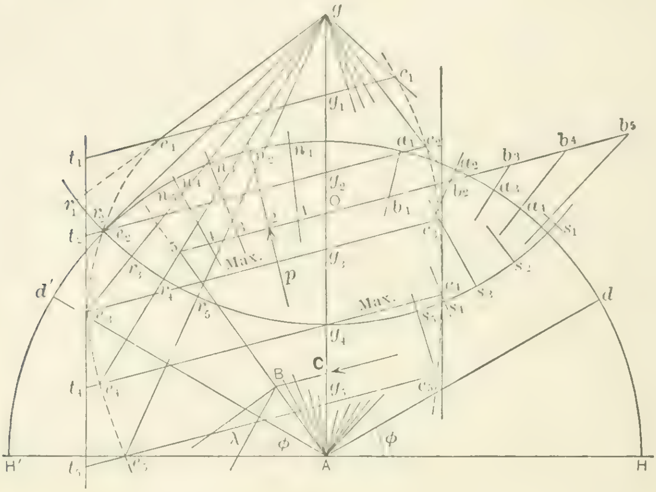

FIG. 10

divided into equal parts; then the weights of the prisms AOb_1, Ab_1b_2, etc., will each equal $\frac{1}{2}\,w.p.\,Ob_1$, where p = length of perpendicular from A upon $5\,Ob_5$. By addition, compute the weights of the trial wedges of rupture AOb_1, AOb_2, etc., and lay off on a vertical Ag to scale, the lengths gg_1, gg_2, etc., to represent these successive weights. Then with A and g as centers and the same radius $= AH = gg_4$, describe the arcs Hda_1 and g_4s_1 and lay off $HAd = \varphi$, $H'AH$ being horizontal. Then, as before, lay off $g_4s_1 = da_1$, $g_4s_2 = da_2$, etc.; whence gs_1, gs_2, . . . , make the angles φ with the normals to planes Ab_1, Ab_2, . . . , and thus give the directions of their reactions. On drawing through g_1, g_2, . . . , lines parallel to $5\,Ob_5$ to intersections c_1, c_2, . . . , with gs_1, gs_2, . . . , the longest of the lines g_1c_1, g_2c_2, . . . , to the scale of force, gives the thrust on AO. It is thus represented by g_4c_4

and it of course acts parallel to $5\,Ob_5$. The plane of rupture on the right of AO corresponding, is Ab_4.

There is likewise a plane of rupture on the left of AO, corresponding to an earth thrust, which for equilibrium must be equal and directly opposed to the thrust on AO just found. The construction, indicated to the left of Ag, is similar to the preceding. The equal parts $O1$, 12, 23, . . . , were each made only half the length of Ob_1, so that the successive trial wedges $AO1$, $AO2$, . . . , are only half the weight of wedges AOb_1, AOb_2, . . . ; but by using double the previous force scale, we can regard gg_1, gg_2, . . . , as representing their weights.

On constructing $H'Ad' = \varphi$ and extending $A1$, $A2$, . . . , to meet arc $Ha_1\,d'H'$ at n_1, n_2, . . . , we next lay off, $g_4r_1 = d'n_1$, $g_4r_2 = d'n_2$, $g_4r_3 = d'n_3$, . . .; then, gr_1 gr_2, gr_3, . . . , make the angles φ with the normals to the planes $A1$, $A2$, $A3$, . . . , and are thus parallel to their reactions, Art. 19.

We now draw through g_1, g_2, . . . , lines parallel to the free surface to intersections e_1, e_2, . . . , with gr_1, gr_2, . . . ; whence by the reasoning of Art. 21, g_3e_3, the maximum intercept of the type g_ie_i, to the scale of loads, represents the active earth thrust on AO. Since the scale of loads for the construction to the left of AO was taken as double that for the construction to the right of AO, the length g_3e_3 should be double the length g_4c_4, which we find to be the case.

The earth thrust on AO, due to the wedge of rupture AOb_4A on the right, acting to the left, parallel to g_4c_4, is thus exactly equal and opposed to the thrust on AO, due to the wedge of rupture $AO3A$ on the left, acting parallel to e_3g_3 and to the right, otherwise equilibrium could not be maintained. Also by Art. 24, the center of pressure of either thrust is on AO at a distance $(1/3)\,AO$ above A.

To find the actual thrusts on the planes $A1$, $A2$, $A4$, $A5$, draw a vertical through e_3 and extend the lines g_1e_1, g_2e_2, g_4e_4, g_5e_5 to meet it at t_1, t_2, t_4, t_5; then the lines gt_1, gt_2, gt_4, gt_5 (not drawn) give the thrusts on the planes $A1$, $A2$, $A4$, $A5$. The proof is easy. Thus the weight of wedge $AO1A$ is gg_1 to scale; the thrust on AO is given by $g_1t_1 = g_3e_3$. The resultant gt_1 of these two forces is the resultant thrust on plane $A1$. Similarly for the other wedges.

The lines gt_1, gt_2, gt_4, gt_5, all lie above the corresponding lines ge_1, ge_2, ge_4, ge_5; hence the actual resultants on the planes $A1$, $A2$, $A4$, $A5$, all make less angles than φ with the normals to these planes; so that stability is everywhere assured. But this would not be the case if we should arbitrarily assume the *direction* of the thrust on AO to lie nearer the vertical than before; since then intersections, $c_1, c_2, \ldots$, would move toward AOg, whereas intersections $e_1, e_2, \ldots$, would move away from Ag; hence the maximum value of g_ic_i $(i = 1, 2, \ldots)$ would have a less horizontal component than before, whereas the maximum value of g_ie_i $(i = 1, 2, \ldots)$ would have a greater horizontal component than was found in Fig. 10. The active thrust from the corresponding wedge of rupture to the right of AO would consequently be less than the thrust pertaining to the wedge of rupture on the left of AO. Equilibrium is thus impossible.

Similar reasoning would show, for an unlimited mass of earth, that, if the direction of the thrust on AO was taken nearer the horizontal than in Fig. 10, equilibrium would be impossible. Hence we have a verification of Rankine's theorem of Art. 11, that the pressure on a vertical plane, in the case of the unlimited mass, subjected to no external force but its own weight, acts parallel to the sloping free surface.

27. Limiting Plane. Suppose the inner face of a retaining wall to occupy the position $A5$, Fig. 10, below the plane of rupture $A3$ on the left of AO. On combining the thrust on AO, acting to the left, parallel to the sloping surface at C, where $AC = (1/3)\,AO$, with the weight of the wedge $OA5$, the resultant on plane $A5$ can be found. Using the left construction, gg_5 is laid off equal to the weight of $AO5 = \frac{1}{2}\,w.p.\,O5$ and $g_5t_5 = g_3e_3$, representing the thrust on AO, is drawn to the left parallel to $5Ob_5$ from g_5 to t_5. Hence a straight line gt_5 (not drawn) to the scale of force gives the thrust on $A5$, *provided* its angle with the normal to $A5$, marked λ in the figure, does not exceed

φ', the angle of friction of earth on wall. It acts at B on $A5$, where $AB = (1/3)\,A5$.

Considering first the case where $\lambda \leqq \varphi'$ and recalling what was proved above, that the resultants on planes $A1$, $A2$, and $A4$ make less angles than φ with the normals to those planes, it is seen that slipping is only impending on the plane of rupture $A3$, if $\lambda < \varphi'$ or on both $A3$ and $A5$, if $\lambda = \varphi'$. When $\lambda < \varphi'$, if the wall moves over at the top, the break will occur only along the plane A3, the earth between this plane and the wall moving over with the wall. If $\lambda = \varphi'$, the slipping will occur along either or both of the planes $A3$, $A5$.

It is convenient to give a name to this plane of rupture on the left of AO ($A3$ in Fig. 10) and it will be called hereafter the *limiting plane* for reasons that will appear as we proceed.

28. *If the inner face of the wall lies above the limiting plane*, the construction pertaining to Fig. 9 applies; so that the thrust on the wall is assumed to make the angle with its normal, φ (or φ' for $\varphi' < \varphi$), the trial prisms of rupture are estimated from AB, Fig. 9, and the thrust determined as shown there.

As was proved in Art. 21, slipping is only impending along the surface of rupture (which is to the right of Ag) and along the wall, and stability is everywhere assured.

When the inner face of the wall is at the limiting plane and $\varphi' \geqq \varphi$, so that the thrust on the wall must be assumed to make the angle φ with its normal, the construction of Fig. 9 leads to the same result as the construction of Fig. 10. This is plain, since the direction of the thrust on the wall by either method is the same, and the amounts must be the same, otherwise the wall reaction by the one method could not balance the thrust as given by the other method. For $\varphi' < \varphi$, see Art. 29.

29. $\lambda > \varphi'$. In case the construction of Fig. 10 gives a thrust on the wall which makes a greater angle with its normal than φ', then the direction of the thrust on the wall must be assumed to make the angle φ' with its normal and the resulting thrust found by the construction of Fig. 9.

If for definiteness we take $A5$ in Fig. 10 as the inner face of the wall and combine its reaction (equal and opposed to the earth thrust thus found) with the weight of earth $AO5$, we find the thrust, acting from left to right, on the plane AO, and for equilibrium this thrust must be opposed and equal to the earth thrust of the mass to the right of the plane AO as found by a construction similar to that to the right of Ag in Fig. 10, only with the direction of the thrust on AO as just found. But this earth thrust on AO acting to left must have a greater horizontal component than the thrust g_4c_4 of Fig. 10, so that when combined with the weight of $AO5$, the resultant on $A5$ will make the angle φ' with the normal to $A5$, and thus lie nearer the horizontal than by the construction of Fig. 10, since it was assumed that $\varphi' < \lambda$.

As a consequence, the equal angles gg_1c_1, gg_2c_2, . . . , gg_5c_5, must increase, for then the points c move away from the vertical Ag and the horizontal component of the maximum value of g_rc_r ($r = 1, 2, \ldots, 5$)—which represents the thrust on AO for the assumed direction—is greater than the horizontal component of g_4c_4. Let such a thrust on $AO = T$. Then the resultant on $A3$, say, is found by combining the weight of wedge $AO3 = gg_3$ (to scale used for wedges to left of AO) with T, represented by a line through g_3, lying above c_3 and having a greater horizontal component than g_3c_3. The resultant on $A3$ is thus represented by a line through g, lying above gc_3 and making a less angle than φ with the normal to $A3$. Similarly it can be shown that the thrusts on the planes $A1$, $A2$, $A4$, lie nearer the horizontal than the positions gl_1, gl_2, gl_4, given in Fig. 10, and thus make angles less than φ with the normals to the planes, so that the conditions for the stability of the granular mass are everywhere assured. The method of proceeding, for the case indicated at the beginning of this article, is thus established. In this case, there is no plane of rupture between AO and $A5$, but slipping is impending along the wall face $A5$ and along the plane of rupture to the right of AO, as found by the construction of Fig. 9.

Since $\lambda < \varphi$, this case, $\lambda > \varphi'$, can only occur when $\varphi' < \varphi$, as when, in the design of walls with a plane inner face, it is arbitrarily assumed that $\varphi' = (1/3)\,\varphi$, say. For stepped walls, where $\varphi' = \varphi$, the case cannot occur.

30. Summary. For all cases of surface slope, when the inner face of the wall is battered, find the limiting plane by the construction of Fig. 10 or preferably by a much simpler method to be given in Art. 65.

(1) When the inner face of the wall makes a less angle with the vertical than the limiting plane, assume the direction of the thrust on it as making the angle φ (or φ' for $\varphi' < \varphi$) with its normal and proceed as in Art. 21, referring to the construction of Fig. 9.

(2) If the inner face of the wall lies below the limiting plane, find by the method of Arts. 26–27 the thrust on the wall and suppose it to make the angle λ with the normal to the wall, Fig. 10. If $\lambda \leq \varphi'$, the thrust on the wall is correctly determined.

By this method, the thrust on a vertical plane passing through the heel of the wall is first found, the thrust having the direction of the sloping surface. This thrust is then combined with the triangular mass of earth lying between the vertical plane mentioned, and the wall, to find the resultant pressure on it.

(3) When $\lambda > \varphi'$; which could only occur when $\varphi' < \varphi$, the direction of the thrust on the wall, will have to be assumed as making the angle φ' with its normal and the construction of Art. 21, Fig. 9, effected to find this thrust.

(4) If the wall leans toward the earth, the construction of Fig. 9 directly applies, the overhang not being too great or when the inner face makes an angle with the vertical not exceeding, say 10°. See Art. 62 for the general case.

31. In the following table are given the values of K, β and γ, corresponding to the thrust, acting parallel to the free surface, on a vertical plane, for the commonly used values of the angle of repose, $\varphi = 30^\circ$ and $\varphi = 33^\circ 41'$.

The free surface is supposed to rise going to the right, as in Fig. 10, and to make an angle i with the horizontal.

The angle made by the plane of rupture to the left of AO, Fig. 10, with the vertical will be called β; the angle made by the plane of rupture on the right of AO with the vertical γ; so that the "limiting plane" makes the angle β with the vertical. Thus in Fig. 10, $\beta = OA3$, $\gamma = OAb_4$.

i	$\varphi = 30°$			$\varphi = 33° 41'$		
	K	β	γ	K	β	γ
0	0.1667	30° 00′	30° 00′	0.1433	28° 10′	28° 10′
5°	.1686	27° 29′	32° 31′	.1447	26° 09′	30° 10′
10°	.1748	24° 50′	35° 10′	.1493	24° 03′	32° 16′
15°	.1865	21° 55′	38° 05′	.1577	21° 46′	34° 33′
20°	.2071	18° 25′	41° 35′	.1718	19° 08′	37° 11′
25°	.2468	13° 38′	46° 22′	.1959	15° 51′	40° 28′
30°	.4330	0°	60° 00′	.2451	10° 59′	45° 20′
33° 41′	......	......	......	.4161	0°	56° 19′

The quantity K is the factor in the formula

$$E = K w h^2$$

and is the same whether the surface rises or falls in going to the right, the thrust E always acting parallel to the free surface. It was computed from Rankine's formula, Art. 48, and checked by the neat construction, Art. 58, Fig. 32.

The values of β and γ were computed very easily from the formulas of Art. 65, and were checked by use of Fig. 37, Art. 65.

From considerations of symmetry, when the free surface $5Ob_5$, Fig. 10, is horizontal or $i = 0$, we must have $\beta = \gamma$; but it requires an analytical demonstration to show, when $i = \varphi$, that $\beta = 0$, $\gamma = 90° - \varphi$, or strictly that as i approaches φ indefinitely, the limiting plane approaches the vertical, and the plane of rupture to the right of AO approaches indefinitely the natural slope.

It may be observed also, what is obvious from the figure, that when the free surface *falls* to the right, β and γ are interchanged. From the large values of γ (or the small values of β) given in the table for values of i near φ, it is seen that the construction of Fig. 10 is impracticable for such large values of i; hence the results of a subsequent chapter are anticipated to aid in immediate practical computations.

Ex. 1. Find, by aid of the table, the Rankine thrust in pounds, $E = K w h^2$, when $i = 20°$, $\varphi = 30°$, $w = 100$ lb. per cu. ft. and $h = 20$ feet.

Ex. 2. Show by aid of a construction similar to that pertaining to the wedges to the right of AO in Fig. 10, that when $i = 0$, $\varphi = 45°$, the coefficient $K = 0.0858$, the thrust being taken horizontal on a vertical plane.

32. Surcharged Wall. Whenever the earth behind a retaining wall extends above its top, the wall is said to be surcharged, and the portion of earth above the level of the top of the wall is called the surcharge. In Fig. 11 is shown a surcharged wall, the surcharge sloping up to b_1, the surface b_1b_4 being then taken horizontal.

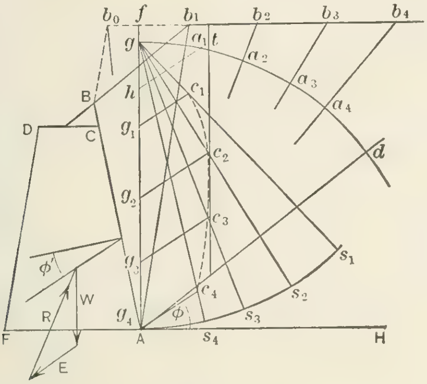

FIG. 11

As usual, one foot length of wall is considered, $ACDF$ being its medial section and B the point where AC meets the sloping surface. The earth thrust E will be assumed to make the angle φ' with the normal to AC, where $\varphi' \leq \varphi$. For convenience in computation, lay off along the level part of the surface the equal lengths, b_1b_2, b_2b_3, . . ., and draw the trial planes of rupture Ab_1, Ab_2, ... On drawing Bb_0 parallel to the base Ab_1, to intersection b_0 with b_4b_1 produced, we have, area triangle $ABb_1 =$ area triangle Ab_0b_1. Also letting $p = Af =$ perpendicular from A on b_0b_4, we have, areas of triangles Ab_0b_1, Ab_1b_2, Ab_2b_3, . . . , equal to $\frac{1}{2}\,p.\,b_0b_1$, $\frac{1}{2}\,p.\,b_1b_2$, . . . ; whence the weights of the successive prisms,

$$ABb_1A,\ ABb_1b_2,\ ABb_1b_3,\ \ldots,$$

are equal to $\frac{1}{2}\, w\, p$ multiplied by

$$b_0b_1,\ b_0b_2,\ b_0b_3,\ \ldots\ .$$

Here, w = weight of 1 cu. ft. of earth in pounds, $p = Af$ in feet, therefore the weights of the successive trial prisms (which are quickly obtained by addition) are given in pounds. We next lay off on a vertical through A, from any convenient point g, gg_1, gg_2, gg_3, . . . , to the force scale, equal to these successive weights.

The construction now proceeds as in Art. 21, Fig. 9. Thus AH being horizontal, construct $HAd = \varphi$ and with A and g as centers and a common radius Ag, describe the arcs ga_1d and As_1. Lay off $As_1 = da_1$, $As_2 = da_2$, . . . , and draw gs_1, gs_2, . . . , which thus make the angles φ with the normals to planes Ab_1, Ab_2, . . . , Art. 19, and give the directions of their reactions. Lines are drawn through g_1, g_2, . . . , parallel to the assumed direction of E, to intersections c_1, c_2, . . . , with gs_1, gs_2, . . . ; whence the longest of the intercepts of the type g_ic_i, which is g_2c_2, to the force scale, gives the thrust E in pounds. Of course other trial planes of rupture must be drawn near Ab_2 and the force triangles corresponding, drawn to obtain the thrust more accurately and it is further advisable to use as large a scale, both for distance and force, as is convenient.

The limiting plane for this figure cannot be accurately found since the direction of the thrust on the vertical plane through A for the unlimited mass is unknown. This direction may be assumed roughly as parallel to Bb_2 (when Ab_2 is the plane of rupture), and the limiting plane found as indicated in Art. 31. As an illustration, suppose in Fig. 11 that Bb_2 makes an angle of 20° with the horizontal and that $\varphi = 30°$; then the table of Art. 31 can be utilized by entering it with $i = 20°$, $\varphi = 30°$, whence $\beta = 18° 25'$. The directions of Art. 30 can then be followed. Thus, if AC lies *above* the limiting plane, the thrust is given correctly by the construction of Fig. 11; if *below* it, then assuming the direction of the thrust on the vertical plane through A as having the direction Bb_2, find its amount by a construction analogous to that of Fig. 11, only the point B is now to be taken where the vertical through A intersects the free surface. The thrust thus found is to be combined with the weight of earth lying between the vertical plane through A and the wall to get the thrust on the wall, provided its obliquity $\lambda \leqq \varphi'$. When $\lambda > \varphi'$, the construction of Fig. 11 directly applies.*

Remark. When the earth contour is irregular or curved, lay off gg_1, gg_2, . . . , equal to the weights of the successive trial wedges of rupture and proceed as before.

Center of Pressure. A simpler construction for the thrust on a surcharged wall will be given in Art. 39, and a resulting computation in Art. 42, to find the center of pressure for various heights of surcharge. See the table giving the results.

33. The disturbing influence of the wall friction in changing the usual distribution of the stresses in an unlimited mass of earth can be illustrated by reference to Fig. 10. Thus if $A5$ is a plane in an unlimited mass of earth with a plane surface, the thrust on the vertical plane AO acts parallel to the surface at C, where $AC = (1/3) AO$ and meets $A5$ at B, where $AB = (1/3) A5$, through which point the resultant weight of $AO5$ also acts. The resultant pressure on the plane $A5$ thus acts at the third point on $A5$ and therefore corresponds to a pressure uniformly increasing from 5 to A, Art. 15.

A similar conclusion holds for any plane, Ab_2 or Ab_5 to the right of AO. Now suppose a wall introduced and that the wall friction causes the thrust on AO to lie above or below the direction first assumed, then the resultant pressure on the plane of rupture, say Ab_4, will not pass through its third point, so that the pressure on the plane, which must be zero at the surface, is no longer uniformly increasing. But at a little distance from the wall, the distance probably increasing with the depth, the usual direction of the thrust in an unlimited mass is doubtless attained and the pressures along any plane are uniformly increasing from the surface down. The inconsistency is due, in part, to having assumed a plane surface of rupture, for the case in question, whereas experiment shows it to be a curved surface. However, for clean sand having little cohesion, the curvature is only slight and the hypothesis of a plane surface should lead to results that are approximately true, which experiments on retaining walls verify.

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*In this last case a simpler test is available. Thus having made the construction, Fig. 11, lay off from g downward to scale, gh, representing the weight of the earth between the vertical plane through A and the wall, and draw from h to the right a (dotted) line to t, equal and parallel to g_2c_2, the reaction E of the wall. Then tg is the thrust on the vertical plane through A. If the angle it makes to the right, with the upward direction of Ag, is greater than the corresponding angle made by Bb_2 with Ag, then, by the reasoning of Art. 29, the construction of Fig. 11 is the true one and slipping will only be impending along Ab_2 and AB. This test avoids the construction of the limiting plane.

The sliding-wedge theory is due to Coulomb and it has been developed by a number of authors whose results vary only as to the direction assumed for the thrust against the wall. Many European authors assume the thrust as making the angle φ with the normal to the wall, thus including the wall friction, whereas other writers follow Rankine in taking the thrust on a vertical plane as parallel to the plane surface assumed. Rankine developed his theory from certain general considerations affecting the relations of stresses in an unlimited mass of earth. In reality he uses an infinitesimal wedge of rupture, as we shall see in Chapter III, and by integration finds the thrust on a vertical plane. For the kind of mass supposed, not subjected to any external force (like wall friction) but its own weight, his results are exact, on the hypothesis of earth endowed with friction but without cohesion. It is only when the application of the Rankine theory to retaining walls is made, ignoring the wall friction, that objection can be made. Two recent authors, Boussinesq and Résal, have endeavored to complete the Rankine theory by considering the influence of the wall friction in modifying it in the vicinity of the wall. In Résal's "Poussée Des Terres," I, the assumption is made that the earth pressures on any plane in a mass of earth increase uniformly in going from the surface downward and that the thrust on a wall has an obliquity φ. After an intricate analysis leading to complicated equations, a method of finite differences has to be eventually invoked to reach numerical results, which are tabulated in excellent shape for use in practice, particularly in the second volume. As the first part of his assumption as to uniformly increasing pressures, although plausible, is open to question, his theory can scarcely claim to be exact. It is, however, the most consistent theory of all, and it is a regret that it is too complicated to present even in outline in an elementary treatise.*

It is evident that in an unlimited mass of earth, the pressures along any inclined plane must increase uniformly from the surface down, as for the case of the vertical plane, Art. 11. The plane surface of rupture leads to such a state of stress, as seen in Fig. 10; hence, for the unlimited mass, the surface of rupture is plane and the assumption leads to exact results. In fact it leads to the Rankine formula, as will be shown in Art. 48.

One other remark may be pertinent. If we suppose a perfectly rigid vertical wall, lying above the limiting plane, on a rigid base, backed by incompressible earth level with its top, then (for these impossible conditions) there will be no wall friction brought into play and the Rankine thrust against the wall will be exerted. If, however, the wall and foundation are elastic, the top will move over, wall friction is experienced and the horizontal component of the thrust becomes less than before, Art. 22. Now a little ways in the interior, the Rankine thrust is still exerted. What becomes of this excess in horizontal thrust? It must necessarily be transmitted to the ground and resisted by friction along the ground surface; which it is generally perfectly capable of doing. The state of stress is complicated and its solution is attempted in Résal's treatise quoted above.

34. Centers of Gravity. In Fig. 12, let the trapezoid $ABCD$ represent the medial section of a wall. Its center of gravity will coincide with the (so called) center of gravity of the trapezoid. Let $DC = a$ and $AB = b$, be the lengths of the parallel sides, whose mid-points are I and N. Then the center of gravity lies on the medial IN. It will now be proved that *if DC be produced to F and BA to E, making $CF = b = AB$ and $EA = a = DC$, then the straight line EF will cut NI at G, the center of gravity of the trapezoid.* Let $\bar{y}$ = perpendicular distance from G to AB, and h = perpendicular distance from C to AB. Divide the figure into two triangles by a line from A to C and take moments about A,

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* One of Résal's conclusions, however, may be stated since it agrees with a theorem otherwise demonstrated above: that when the inner face of the wall lies at or below the "limiting plane," the thrust on it is to be found as explained in Art. 27, taking $\varphi' = \varphi$. Thus Fig. 10, if the wall $A5$ lies at or below the plane of rupture $A3$, the thrust on AO acting parallel to the surface is found and then combined with the weight of earth $AO5$ to obtain the thrust on $A5$.

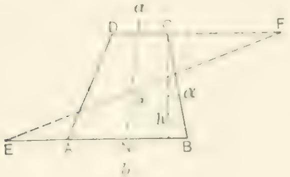

FIG. 12

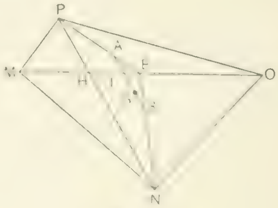

FIG. 13

$$\therefore \frac{a+b}{2} h \bar{y} = \frac{bh}{2} \frac{h}{3} + \frac{ah}{2} \frac{2}{3} h$$

$$\therefore \bar{y} = \frac{(b+2a)\,h}{3(a+b)} .$$

Now if G is the center of gravity, lying somewhere on NI, we have,

$$\frac{\bar{y}}{h} = \frac{NG}{NI} = \frac{b+2a}{3a+3b} .$$

By geometric division,

$$\frac{NG}{NI - NG} = \frac{NG}{GI} = \frac{b+2a}{2b+a} = \frac{b/2+a}{b+a/2} = \frac{EN}{IF} ,$$

since we laid off, $EA = a$ and $CF = b$.

Hence from the equality of the second and fifth ratios, it is seen that EF cuts NI at G, the center of gravity of the trapezoid, the two triangles ENG and GIF being similar and giving the above proportion. The construction indicated above is thus justified.

Regarding AB as horizontal, let $\bar{x}$ represent the horizontal distance from A, Fig. 12, to the vertical through the center of gravity G of the trapezoid, when CB is inclined at an angle α with the vertical. Dividing $ABCD$ into the two triangles ADC and ACB and taking moments about A, we derive after reduction,

$$\bar{x} = \frac{2b^2 + 2ab - a^2 - (2a + b)\, h \tan \alpha}{3\,(a + b)}$$

When the projection of C on AB falls to the right of B, change the sign of $\tan \alpha$.

The center of gravity of any quadrilateral $MNOP$, Fig. 13, can be found as follows: *Let E be the mid-point of MO,* $EA = 1/3\ EP$, $EB = 1/3\ EN$; *join B and A with a straight line and let it intersect MO at I; if now on BA, BG is laid off equal to IA, then G is the center of gravity of the quadrilateral MNOP.* Noting that A and B are the centers of gravity of the triangles MOP and MNO and that $PN \parallel AB$,

$$\frac{BI}{IA} = \frac{NH}{HP} = \frac{\text{area } MNO}{\text{area } MOP} = \frac{GA}{GB},$$

G being regarded as the position of the resultant of two forces proportional to areas MNO, MOP, and applied at B and A respectively.

By composition,

$$\frac{BI + IA}{IA} = \frac{BG + GA}{BG} \quad \therefore\ BG = IA.$$

This shows that the construction indicated is valid. It is often the best construction for the trapezoid, the lines to be drawn all lying within the figure.

35. Test of the Stability of Retaining Walls.

Ex. 1. Consider a wall of the trapezoidal type, Fig. 12, backed by earth sloping upward from C at an angle of 6° with the horizontal, the earth weighing 100 and the wall 140 lb. per cu. ft. Let $a = 2'$, $b = 10'$, $h = 20'$ and $\alpha = 17°\ 21'$, whence horizontal projection of $CB = h \tan a = 6.25'$. By the formula for $\bar{x}$ or the construction of Art. 34, we find $\bar{x} = 4.12'$.

The earth thrust against CB for $\varphi = 33^\circ 41'$, has been found, Ex. 1, Art. 23, to be $E = 9550$ lbs. It acts at (1/3) BC above B on BC and makes an angle $\varphi = 33^\circ 41'$ with the normal to BC. Hence, as in Fig. 11, lay off this direction at the point of action on the wall and extend it to meet the vertical through the center of gravity of the wall, which passes $\bar{x} = 4.12'$ to the right of the outer toe. At the point of intersection, combine the weight of the wall, $W = 16{,}800$ lbs. acting vertically downward, with the earth thrust $E = 9550$ lbs., and find that the resultant cuts the base 3.6′ from the outer toe or 1.4′ to the left of its center. It is often best to combine the forces as illustrated in Fig. 6, if thereby a larger scale can be used.

The resultant on the base is found to make an angle of 14° 09′ with the vertical; hence, by Art. 16, the factor of safety against sliding is,

$$\frac{\tan\theta''}{\tan 14^\circ 09'} = \frac{\tan\theta''}{.25},$$

where θ'' is the angle of friction of masonry on earth. If $\tan\theta''$ is taken as 0.33 for moist or wet clay and 0.50 for dry clay, the factors of safety are, 1.3 and 2 respectively.

The vertical component of the resultant on the base is found to be $F = 24{,}200$ lbs.; hence the soil reactions at the toes are, by formula (3) of Art. 15,

$$p = \frac{F}{l}\left(1 \pm \frac{6a}{l}\right) = \frac{24{,}200}{10}\left(1 \pm \frac{6 \times 1.4}{10}\right)$$

or 4450 lb. per sq. ft. at the outer toe and 390 lb. per sq. ft. at the inner toe, which are safe values for dry sand or dry clay. The wall is thus stable for a fairly good earth foundation and the entire base is under compression. Since, in this example, $\alpha = 17^\circ 21'$ and, from the table in Art. 31, $\beta > 26^\circ$, BC in Fig. 12 lies above the "limiting plane" and the earth thrust is correctly estimated by the construction of Fig. 9, where the full friction between the earth and wall is allowed.

Ex. 2. Investigation of the stability of the wall of Ex. 1 by the Rankine method. The wall $ABCD$ is drawn to scale in Fig. 14 (the original scale was 2 ft. to 1 in.). $AB = 10'$, $CD = 2'$, height of wall $BL = 20'$, $CL = 6.25'$, vertical $BH = 20.65$ ft.

The earth thrust on the vertical plane BH, from the inner toe to intersection with the free surface CH, can be found by a construction similar to that of Fig. 10 (using either the construction to the right of AO or the one to its left) and is found to be $E = 6200$ lbs., this thrust acting at 1/3 BH above B, parallel to HC. Its value may be checked by aid of the tabular results in Art. 31, where, for $i = 6^\circ$, $\varphi = 33^\circ 41'$, we find by interpolation, $K = 0.145$; whence,

$$E = Kw.\,BH^2 = 0.145 \times 100 \times (20.65)^2 = 6180 \text{ lbs.}$$

This thrust will now be combined with the combined weight of the wall and of the earth BCH. To find x, the horizontal distance from A to the vertical through the center of gravity of these weights, take moments about A.

The weight of earth $BCH = \frac{1}{2} \times 100 \times 20.6 \times 6.25 = 6436$ lbs. and it acts 1/3(6.25) = 2.08′ from BH or 7.92′ from A; hence its moment about

A is 6436 × 7.92 = 50,973 ft. lbs. From Ex. 1, the moment of the weight of wall, 16,800 lbs., about A is, 16,800 × 4.12 = 69,216 ft. lbs. The sum of these moments, 120,189, divided by the weight of wall and earth, 23,236, gives $x = 5.18$ ft. Hence draw a vertical 5.18′ to right of A to meet E, the earth thrust on BH, at G. Lay off, to scale, $E = GM = 6200$ lbs., GM being parallel to the surface and passing through the point on BH, (1/3) BH above B. At M, lay off MN vertically equal to 23,236 lbs.; extend the resultant NG to the base at I. AI is found to be 3.5 ft., or nearly the same as before. The point I is still within the middle third of AB, so that the whole of the base is under compression and the conclusions pertaining to Ex. 1 obtain here.

The reason the two solutions, one by the wall friction method, the other after Rankine, so nearly agree for the wall examined, is because the thrust E on BH, by the Rankine method, is combined with such a large weight of earth BCH, to get the thrust on BC. The resultant on BC thus makes nearly the same angle with the normal to BC as in the previous solution. But when BC is vertical, the two theories differ widely, the first assuming the thrust on the wall to make an angle φ (when $\varphi' = \varphi$) with the horizontal, whereas Rankine assumes the thrust on it, to act parallel to CH.

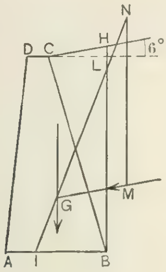

FIG. 14

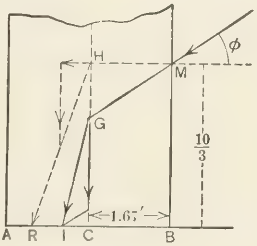

FIG. 15

Ex. 3. Let Fig. 15 represent a portion of a vertical brick wall weighing 125 lb. per cu. ft., the wall being 10′ high and 10/3 = 3.33′ thick and backed by earth level with its top, the earth weighing 100 lb. per cu. ft. Assume $\varphi = \varphi' = 33° 41'$; then by the method of Art. 21, including all the wall friction, we find $E = K w h^2 = 0.1304 \times 100 \times 10^2 = 1300$ lbs.

This thrust acts at M, Fig. 15, 3.33′ above B and is inclined to the normal to BM at the angle φ. Extend it to meet the vertical through the center of the base C at G and combine it at G with the weight of the wall, 4167 lbs., which acts along GC. The resultant GI cuts the base at I, 0.5 ft. to the left of its center and 0.06 ft. within the middle third limit.

Ex. 4. Solve the same example by the Rankine method. The thrust E at M is now supposed to act horizontally. Its amount, using $K = 0.1433$ from Art. 31, is $0.1433 \times 100 \times 10^2 = 1433$ lbs. At H, where it meets the vertical CH, combine with the weight of the wall. The resultant (dotted line) is found to cut the base at R, 0.52 ft. from A, outside of the middle third, so that only 3 × 0.52 = 1.56 ft.—less than one-half the base—is under stress.

If the Rankine theory was applicable, the wall should at once be declared unsafe. As a matter of fact it is entirely safe, for Sir Benjamin Baker tells us* that hundreds of brick revetments have been built by the Royal Engineer officers with a thickness of only $0.32h$ for a vertical wall, which are stable.

Rankine states,† in effect, that (using his theory) in practice, the resultant on the base is allowed to cut it 1/8 the thickness from its outer toe by British engineers and 1/5 the thickness by French engineers. It would follow that only 3/8 of the base should be under compression in the first instance and only 3/5 in the second, Art. 15.

No stronger indictment could be brought against the application of his theory to vertical rectangular retaining walls than Rankine himself furnishes us, for it is unreasonable to suppose that walls that have been stable for years should bear only upon about half their base.

The defect in the Rankine theory when applied to rectangular walls backed with level-topped earth is that all the wall friction is ignored. It leads to exaggerated dimensions when the resultant on the base is required to pass within its middle third. A certain amount of friction between the earth and wall is *always* exerted. It is only a question of how much should be allowed in *designing* a wall. In Art. 14, suggestions are made along this line, which are carried out in the design of walls, Chapter IV.

These suggestions refer particularly to walls with plane inner faces, where it was feared that the lubrication of the wall by water might diminish materially, at the time of heavy rains, the value of φ'. But when the rear face is stepped, the surface of impending slipping will be a plane in the earth, touching the edges of the steps, and there seems no more reason why the coefficient of friction ($\tan \varphi$) should diminish along this plane than along the surface of rupture. For walls exposed to no vibration, the full friction of earth on earth can be allowed for the stepped walls, but where vibration is excessive, the suggestion of Art. 14 may be carried out or the judgment of the engineer invoked to arbitrarily increase the thickness of the wall over computed values.

In this article, the thrusts are estimated for usual conditions (no factor of safety having been introduced) which doubtless obtain nearly all the time. The complete design requires consideration, too, of the exceptional conditions, as to which see Chapter IV.

Ex. 5. Let us assume again the data of Ex. 1, only with the surface CH sloping at the angle of repose and with the position of the point A to be determined, so that the resultant on the base AB shall cut it $(1/3)\ AB$ from A, Fig. 16, p. 54. By Art. 31, the "limiting plane" coincides with BH, so that the Rankine method outlined in Ex. 2 is applicable and is the only correct one to follow.

Find by measurement on a large scale drawing or by computation, $BH = 24.17$. From the table, Art. 31, $K = 0.416$ ∴ the thrust on BH, acting parallel to CH, is $E = K w h^2 = 0.416 \times 100 \times (24.17)^2 = 24{,}300$ lbs. Draw a vertical through D meeting AB at V, whence $VB = 8.25$ ft.; also show, as in Ex. 2, that the combined weight of the portion of the wall *VBCD* and the earth *BCH* is 21,900 lbs. and that its resultant acts through 1, 4.02′ to right of *V*.

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* *Minutes of Proceedings*, Inst. C. E., Vol. LXV, p. 140; reprinted in Van Nostrand's Science Series, No. 56.

† Applied Mechanics, p. 227.

The thrust *E* acts on *BH*, (1/3) *BH* above *B*. Lay off *OP* parallel to *CH* to represent this thrust and draw *PQ* vertically equal to the combined weight, 21,900 lbs., *OQ* being their resultant. Produce *E* to meet the vertical through the center of gravity of *VBCD* and *CBH* at 1 and draw 12 parallel to *OQ* to give the position of this resultant, *OQ*. We now resort to the method of trial and error to fix the position of *A*. *AV* was first assumed 4 ft. and, by a construction similar to the one that follows, was found to be too large. Taking $AV = 3$ ft. we find weight of $ADV = \frac{1}{2} \times 20 \times 3 \times 140 = 4200$ lbs. and it acts along the vertical, $(\frac{1}{3})\ AV = 1$ ft. to the left of *V*. This vertical meets the line of action 12 of the resultant whose magnitude is *OQ* at 2. At this point combine the two forces. On laying off *QR* = weight of *ADV*, *OR* represents the magnitude and direction of the final resultant on the base. Its position is found by drawing a line through 2 parallel to *OR*. It meets *AB*, 3.83 ft. to the right of *A*. Since $AB/3 = 11.25/3 = 3.75$ ft., it is seen that the resultant passes 0.08 ft. to the right of the outer third point or slightly inside the middle third of the base. The requirement originally stated is practically fulfilled, though another trial might prove instructive to the student as showing the ease of working by this tentative method.

Note. In practice, walls are usually designed either for a horizontal free surface or for one sloping at the angle of repose. Supposing $\varphi' = \varphi$ (always true for a stepped inner face), then as shown in Ex. 5, for the latter case, the Rankine method gives the true solution; but for vertical walls with the earth surface horizontal, a comparison of the results of Exs. 3 and 4 shows that the Rankine method is inapplicable and leads to exaggerated dimensions. However, when the inner face of the wall has a considerable batter, as usual, it is seen from the results of Exs. 1 and 2, for earth surface level or nearly so, that the Rankine method nearly agrees with the wall-friction method and leads to a slightly greater thickness at the base, which is on the side of safety. For reinforced T walls or those with counterforts, where the earth thrust has to be estimated on a vertical plane a little distance from the vertical slab, the Rankine method is again applicable; so that in practical design, the Rankine thrust is called for in the majority of cases. On account of its theoretical and practical importance, it will be considered in detail in a subsequent chapter.

It is well at this stage for the student to refer to the results of experiments on model retaining walls, given in Appendix II, as they will tend to give him some confidence in the theories advanced above.

36. Passive Thrust against a Wall. See Art. 20. In Fig. 17, let *Ag* represent the vertical inner face of a wall, the earth to the right having a plane surface *g*123 . . . , and suppose the wall subjected to a force acting to the right so great that motion is impending of a wedge of earth *up* some plane as *A*2, *A*3, etc., and *up* the wall; then the frictional components of the reactions of both plane and wall will act *down* or opposed to the impending motion. The direction of the reaction of the wall will then be, say g_2c_2, inclined below the normal to Ag at the angle φ' (or φ if $\varphi' > \varphi$), and the reaction of the plane of rupture will lie nearer the horizontal than its normal and will make the angle φ with it.

For convenience, lay off along the free surface, $g1 = 12 = 23$, etc., compute the weights of the trial wedges of rupture, $Ag1$, $Ag2$, . . . , and lay off, to the scale of force, gg_1, gg_2, . . . , equal to these weights. (Any weight, as that of $Ag2$, is found by multiplying $\frac{1}{2} w. \overline{g2}$ by the perpendicular from A upon $g2$.) The next step is to draw two arcs of circles with A and g as centers and a common radius Ag and lay off the angle $Ags = \varphi$ and mark the intersections of $A1$, $A2$, . . . , with the arc ag, a_1, a_2, ... Then to find the directions of the reactions of the successive trial planes of rupture, let us first consider one of them as $A2$. Draw Aa horizontal and let α = angle made with the vertical Ag by the normal to $A2$ $\therefore \alpha = a\,A\,a_2$. The direction of the reaction gs_2 of the plane $A2$ must make an angle φ above its normal,

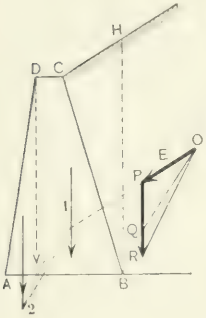

FIG. 16

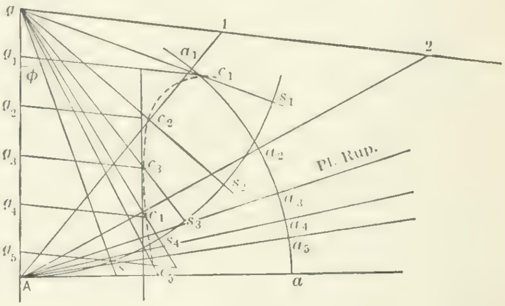

FIG. 17

$$\therefore\ Ags + sgs_2 = Ags_2 = \varphi + \alpha = Ags + a\,Aa_2;$$

whence $sgs_2 = a\,A\,a_2$ or chord ss_2 = chord aa_2. A similar conclusion holds for any of the planes $A1$, $A2$, . . . , so that to find the directions of their reactions, lay off $ss_1 = aa_1$, $ss_2 = aa_2$,

$ss_3 = aa_3$, . . . , and draw gs_1, gs_2, gs_3, . . . , to give the directions sought.

To complete the triangle of forces for each trial wedge of rupture, draw lines parallel to the wall reaction, through g_1, g_2, . . . , to intersections c_1, c_2, . . . , with gs_1, gs_2, . . . ; then the *least* of the intercepts g_3c_3 is approximately the *passive thrust* and $A3$ the corresponding plane of rupture. This follows, because if the thrust is any greater than g_3c_3, c_3 moves to the right and the new gc_3, representing the reaction of $A3$, lies above the old position and thus makes a greater angle than φ with the normal to $A3$, which is inconsistent with stability, Art. 5.

The passive thrust (or resistance) is much larger than the active thrust and increases with φ and φ'. To keep the figure within bounds the values of φ and φ' were taken small. When the direction of the thrust on the vertical plane Ag is taken parallel to the surface, the resulting passive thrust equals that on a vertical plane in an unlimited mass of earth.

For passive thrust, the plane of rupture is much nearer the horizontal than for active thrust. The surface of rupture being longer, is doubtless more irregular and further removed from a plane than for the case of active thrust, so that the theory is not so close to reality as in the latter case. The construction of Fig. 17 can readily be extended to battered walls and various slopes and surcharges of earth, but the matter is not of sufficient practical importance to enter into further.

The *center of pressure* of the passive thrust is on Ag at a distance $(1/3)\ Ag$ above A.

CHAPTER III

NON-COHESIVE EARTH. ANALYTICAL METHODS

37. Active Thrust. In the analytical method that will now be given, trial wedges of rupture will be assumed, each one in turn being treated as a true one until the application of the principle of Art. 5 enables us to select the only true one and to find the corresponding thrust.

In Fig. 18, let AF represent the inner face of the wall and $FPQRC$ the earth surface, the length of wall considered being 1 foot, perpendicular to the plane of the paper. The wall AF will be supposed to resist by a reaction E, inclined at the angle φ' (or φ when $\varphi' > \varphi$) to the normal to the wall, the tendency of the earth to slide down some plane AC, passing through A. Let G = weight of earth $AFPQRC$, contained between two planes perpendicular to the inner face of the wall and 1 foot apart, and S = reaction along plane AC, inclined at the angle φ to the normal to AC, the component of S parallel to AC, acting up along AC. Also, let AC make the angle γ with the horizontal AH and the angle ϖ with the vertical. Finally let α = angle

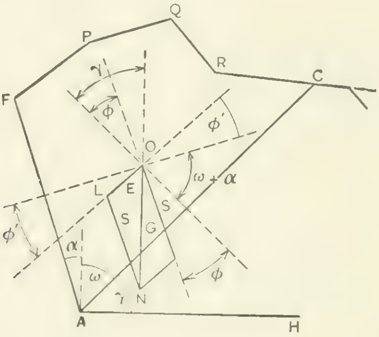

FIG. 18

made by AF with the vertical. The values of certain angles are shown on the figure.

The three forces E, G and S are in equilibrium and may be represented in magnitude and direction by the sides LO, ON, NL of the triangle ONL, E and S acting up and G down. Hence noting that $ONL = \gamma - \varphi$ and that $NLO = \varphi + \tilde{\omega} + \alpha + \varphi'$, we have, by the law of sines,

$$\frac{E}{G} = \frac{\sin ONL}{\sin NLO} = \frac{\sin(\gamma - \varphi)}{\sin(\varphi + \tilde{\omega} + \alpha + \varphi')} \quad . \quad . \quad (5)$$

As AC varies its position, the true value of E is the maximum obtained by the construction above for the various trial wedges of rupture; for if AC is the plane of rupture corresponding to this greatest trial reaction E, if E is taken less than this, the point L will lie nearer O and NL will lie nearer the vertical, hence NL or S will make an angle greater than φ with the normal to AC, which, Art. 5, is inconsistent with the laws of stability of a granular mass. Hence the greatest of the trial thrusts is the least thrust consistent with equilibrium, and this value of E is the magnitude of the actual *active* thrust exerted against the wall, caused by the impending motion of the wedge $AFPQRC$ down the plane AC.

38. Let us now consider the simpler earth profile $FRPQ$ of Fig. 19 and assume that the true plane of rupture AC intersects the part RP of the profile. If FB is drawn parallel to a line from A to R to meet RP produced at B, then the areas of the two triangles AFR and ABR are equal and area $AFRC$ is equal to that of the triangle ABC.* Hence dropping a perpendicular AT from A upon BC, we have the weight of the wedge $AFRC$, =

$G = \frac{1}{2} w . AT.BC$, where w = weight of a cubic foot of earth

in pounds, the foot being the unit of length. In Fig. 19, draw the line CI to intersection I with the line of natural slope AD, making the angle $ACI = (\varpi + \varphi + \varphi' + \alpha)$. Then, noting that $(\gamma - \varphi) = CAJ - DAJ = CAI$, we can replace the ratio of the sines in (5) by CI/AI, so that (5) can be written, on substituting the above value of G,

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* In Fig. 18, the point B is found on the prolongation of CR, so that area $AFPQRC$ is equal to area triangle ABC. All the subsequent conclusions pertaining to Fig. 19 apply without change to Fig. 18, with the point B thus found.

$$E = \frac{1}{2} w.\ AT.\ BC.\ \frac{CI}{AI} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

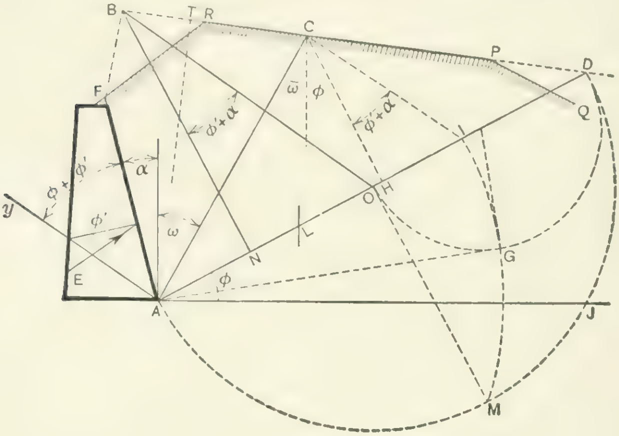

FIG. 19

On drawing BO parallel to CI to intersection O with AD, we have from the similar triangles BOD and CID,

$$BC = BD\frac{OI}{OD};\ CI = ID\frac{BO}{OD};$$

whence (6) takes the form,

$$E = \frac{1}{2} w \left(\frac{AT.\ BD.\ BO}{OD^2}\right) . \frac{OI.ID}{AI} \quad . \quad . \quad . \qquad (7)$$

The terms in ( ) may be reduced to a simpler form on noting that $AT.BD$ represents double the area of the triangle ABD and can be replaced by $AD.BN = AD.BO \cos OBN$, where BN is perpendicular to AD; whence,

$$E = \frac{1}{2} w.\ AD \cos OBN \left(\frac{BO}{OD}\right)^2 . \frac{OI.ID}{AI} \quad . \quad . \qquad (8)$$

On drawing CH perpendicular to AD, we note that AC makes the angle ϖ with a vertical at C and that CH makes the angle φ with the same vertical, the sides being perpendicular to those of the angle $DAJ = \varphi$. Hence since ACI was laid off $=$ $(\varpi + \varphi + \varphi' + \alpha)$, it follows that $HCI = NBO = (\varphi' + \alpha)$. This angle being constant, BO and OD are constant as AC varies in position, so that the only variable part of (8) is the last fraction. For brevity write $AI = x$, $AD = a$, $AO = b$, then the variable term can be written,

$$\frac{OI.ID}{AI} = \frac{(x - b)\,(a - x)}{x} = a + b - \frac{ab}{x} - x$$

which is a maximum for $x = AI = \sqrt{ab}$. Substituting this value of x in the variable term, it reduces to,

$$a + b - 2\sqrt{ab} = \frac{(a - \sqrt{ab})^2}{a} = \frac{\overline{ID}^2}{AD},$$

the point I now corresponding to the position of C for the true plane of rupture AC.

On substituting this maximum value for the variable term and noting from the similar triangles, BOD, CID, that $\frac{BO}{OD} = \frac{CI}{ID}$, (8) reduces to,

$$E = \frac{1}{2} w . \overline{CI}^2 \cos HCI = \frac{1}{2} w . IL . CH \quad . \quad . \quad (9)$$

where IL has been laid off along IA equal to CI. If a line is drawn from L to C, it is seen that *the active thrust E against the wall in pounds is exactly represented by the area of the triangle ILC multiplied by w, the weight per cubic foot of the earth in pounds.**

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* This conclusion, for the case of a plane earth surface, was reached by Rebhann in 1871, and later by Weyrauch, both proceeding along independent lines, different from the above.

Since the angle $HCI = (\varphi' + \alpha)$, the formula can likewise be written,

$$E = \frac{1}{2} w . \overline{CI}^2 \cos (\varphi' + \alpha) \quad . \quad . \quad . \quad . \qquad (10)$$

All the preceding results refer to the inner face of the wall AF lying to the left of the vertical through A. In case the wall leans toward the earth or AF lies to the right of the vertical through A, all the preceding formulas hold on simply changing α to $(-\alpha)$, as will be evident on reviewing the demonstration.*

If we draw Ay parallel to CI (or to BO), since CI makes the angle $(\varphi + \varphi' + \alpha)$ with the vertical, it follows that Ay makes the same angle with the vertical, hence $FAy = \varphi + \varphi'$. The same conclusion holds for leaning walls, where α is changed to $(-\alpha)$.* It is tacitly assumed above that AF lies above the limiting plane and that the conditions outlined in Art. 28 or 32 are satisfied.

39. To find the earth thrust E against the plane AF, whether AF lies to the left or to the right of the vertical through A (in the latter case, α is not to exceed about 10°), first find the point B, so that area triangle ABR = area triangle AFR and draw the line Ay making the angle $(\varphi + \varphi')$ with AF, as shown. Then draw BO parallel to Ay to intersection O with AD, inclined at the angle φ to the horizontal. Let AD intersect RP, produced if necessary, at D. Then, $x = AI = \sqrt{AD . AO}$ can either be computed, on measuring AD and AO to scale, otherwise two constructions, given by geometry, are available for locating I.

By the first, a semicircle is described on AD as a diameter and at the point O, a perpendicular is erected to AD, meeting the semicircle in M; then AI is laid off equal to the chord AM. By the second construction, a semicircle is described on OD as a diameter, a tangent to it, AG, from A is drawn, limited by the radius perpendicular to it, and finally AI is laid off equal to AG.

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* See Arts. 19 and 62 as to the direction of the thrust to assume for leaning walls.

The point I having been located by calculation or by either of the constructions, IC is drawn parallel to Ay to intersection C with RPD; whence AC is the plane of rupture.

On measuring the length CI, also the length of the perpendicular CH, from C to AD, to the scale of distance, the earth thrust E on AF is given by $\frac{1}{2} w.\ CI.CH$; otherwise E equals w times the area of the triangle ICL, where $IL = CI$. Where the dimensions are in feet and w in pounds per cubic foot, the thrust E is given in pounds. The same construction applies to a more irregular earth contour, as in Fig. 18, as soon as the point B is found. In any case, if AC is not found to intersect the free surface on the plane assumed, another plane may be tried, etc. For a curved earth contour, the general method of the preceding chapter can be used.

40. When the free surface is plane and either rises or falls going from the wall or is horizontal, the construction is simplified since the point B then coincides with F, which generally falls at the inner upper corner of the wall. It may lie above it, as in Fig. 19, or it may fall below it. In any case, AF is the area pressed along the inner face of the wall, extended if necessary.

Fig. 20 shows the construction for this common case, AB being the inner face of the wall. The letters having the same significance as before, the thrust $E = w \times$ area triangle CIL.

Ex. 1. Find the value of E pertaining to Ex. 1, Art. 35, by this construction.

Ex. 2. Find the value of E for some of the examples of Art. 23.

The construction is a good one except when angle ICD is small, when the point C cannot be located very accurately. In that case, use the methods of Art. 21 or perhaps the construction, Fig. 22. Both the graphical methods fail when the inclination of BD to the horizontal is large, on account of the intersection D being too remote.

In that case, formulas will have to be resorted to. In Fig. 20, as BD approaches the natural slope or approaches parallelism to AD, the distance AD increases indefinitely and so does $AI = \sqrt{AD.AO}$, so that the triangle CIL approaches in size the similarly constructed triangle with I anywhere on AD and $IC \parallel Ay$, meeting a line through B parallel to AD at C. The area of the triangle CIL corresponding, multiplied by w gives the thrust E when BCD is at the natural slope. Also regarding now C as in true position, CI being finite, since AC and AI approach infinity with AD, it follows that the plane of rupture AC approaches indefinitely AD or the natural slope as AD indefinitely increases or as BD approaches indefinitely the natural slope.

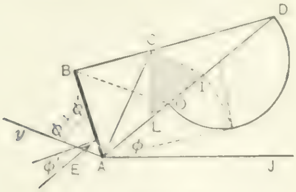

FIG. 20

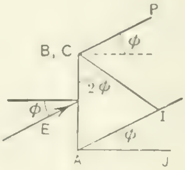

FIG. 21.

As a special case, take AB vertical and let C coincide with B, Fig. 21, the earth surface, CP being at the natural slope. Then if E makes the angle $\varphi' = \varphi$ with the normal to AB, $\angle\ ACI = \varphi + \varphi' = 2\varphi$; whence $\angle\ AIC = (90 - 2\varphi) + \varphi = 90 - \varphi = \angle\ IAC \therefore CI = AB = h$. $\therefore$ since $\alpha = 0$, by (10),

$$E = \frac{1}{2} w\, \overline{CI}^2 \cos\varphi = \frac{1}{2} w\, h^2 \cos\varphi \quad . \quad . \quad . \quad (11),$$

the well known formula for the thrust on a vertical plane in an unlimited mass of earth with the surface sloping at the angle of repose. This formula gives a large excess to the true thrust in any practical case, since an infinite plane of rupture is never realized for any possible back filling. See Art. 43, Fig. 23.

Ex. 1. Find the thrust against the vertical plane BH, Fig. 16 in Ex. 5, Art. 35.

Ex. 2. Check some of the values of K in Art. 31.

41. Several interesting conclusions follow from the construction, Fig. 20. Observing that if a line is drawn from the center of the semicircle parallel to IC, it meets BD at its mid-point, it is seen that C is to the left of this mid-point, but approaches it as BD approaches BO, since when OD is very small, the point I and the center of the semicircle are nearly coincident.

At the limit, when BD falls upon the fixed line BO, C is exactly at the mid-point of $BD = BO$.

When BD falls below BO, D lies below O, but the semicircle is described on DO as diameter, since $AI = \sqrt{AD.AO}$, as before; but IC, drawn parallel to BO, now meets BD to the right of its mid-point.

In Fig. 22 is given another construction for locating the points C and I. Extend the line DB to meet the line Ay at y; then from similar triangles,

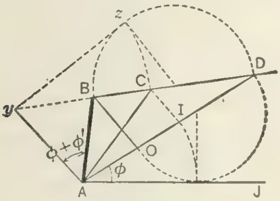

FIG. 22

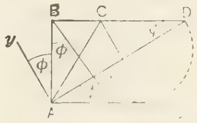

FIG. 22a

$$\frac{AI}{yC} = \frac{AO}{yB} = \frac{AD}{yD} = m, \text{ say,}$$

on substituting, $AI = m \,.\, yC$, etc., in the relation previously established,

$$AI^2 = AO.AD$$

and striking out m^2, we have,

$$yC^2 = yB.yD.$$

On describing a semi-circle on BD as diameter, drawing yz (limited by a radius perpendicular to it) tangent to the semicircle, then from the last equation, yC is to be laid off equal to yz to locate the point C.*

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* See this construction in Prelini's "Earth Slopes, Retaining Walls and Dams," where Rebhann's neat analysis is given.

The plane of rupture AC is thus determined more directly than by the preceding construction. On drawing CI parallel to Ay to intersection I with AD, the thrust $E = \frac{1}{2} w . CI \times$ perpendicular from C on AD, as given by eq. (9).

It will now be proved, *for a perfectly smooth vertical wall* ($\varphi' = 0$, $\alpha = 0$), *with earth surface horizontal, that the plane of rupture bisects the angle between the vertical and the line of natural slope.* In Fig. 22 (*a*), conceive AB to be vertical and BD horizontal; then, since $\varphi' = 0$, $ABO = \varphi$, and since BD is parallel to AJ, $BDA = \varphi$. Hence the two right triangles, ABO, ABD, are similar,

$$\therefore \frac{AO}{AB} = \frac{AB}{AD} \therefore AB^2 = AO . AD = AI^2.$$

It follows that $AB = AI$, whence the two right triangles BAC and CAI are equal

$$\therefore \angle BAC = \angle CAI = \frac{1}{2}(90^\circ - \varphi),$$

which was to be proved.

The horizontal thrust against the vertical wall AB is,

$$E = \frac{1}{2} w . CI^2 = \frac{1}{2} w . BC^2 = \frac{1}{2} w . AB^2 \tan^2\left(45^\circ - \frac{\varphi}{2}\right) . \quad . \ (12)$$

This is the well known expression for the horizontal thrust against a vertical plane, in an unlimited mass of earth with a horizontal surface.*

42. Historical Note. The sliding-wedge theory of earth pressure was first put upon a scientific basis by Coulomb (1773). He assumed $\varphi' = 0$, whereas Poncelet (1840) took φ' at its true value and developed in an elegant manner a graphical method equivalent in its results to the above. Since Poncelet's time, a host of authors have developed the sliding-wedge theory along more or less original lines. In particular, Rebhann in 1871, deduced formula (9) of Art. 38 with the corresponding construction and later. Weyrauch, by an independent analysis, reached similar conclusions, the direction of the thrust on the wall, however, not being assumed, but forming part of the problem.

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* In the author's "Practical Designing of Retaining Walls" (Van Nostrand's Science Series), pp. 90–93, is given a very simple diagram for finding the Rankine thrust on a vertical plane, the surface sloping at any angle, and the plane of rupture (limiting plane) corresponding, the analysis being similar to that which precedes. It is omitted here, because the same subjects will be fully considered, from another standpoint, further on.

Rankine in 1856 from considerations of certain properties of stress (really affecting an infinitesimal wedge of a granular material), put the theory of earth pressure on a vertical plane in an unlimited mass of earth, on a thorough and exact scientific basis. It is only in his proposed application of this theory to rough retaining walls that criticism can be made. Boussinesq attempted to complete the Rankine theory by considering the influence of the wall friction in modifying the Rankine thrust. His analysis only leads, in a simple case, to two limits to the thrust, whose most *probable* value is the smaller of these limits augmented by $9/22$ of their difference. The work of Résal (1903) has already been alluded to in Art. 33.

The author has been careful to point out in Arts. 26–30 where Rankine's theory applied to retaining walls gives exact results and where inexact results. In the latter case, the sliding-wedge theory is resorted to, for though it is admittedly approximative for such cases, it is easily understood and answers the needs of practice in ordinary cases. The sliding-wedge theory gives exact results for the case of the unlimited mass of earth and, in fact, leads to the Rankine formula, Art. 48.

43. Center of Pressure. Referring to Fig. 20, it is seen that, for the same surface slope and inclination of AB, if the vertical height h of AB varies, then CI (always parallel to Ay) will vary directly with AB or with h. Hence the thrust E, which, by (9), varies as $\overline{CI}^2$, will vary as h^2 and can be represented by an expression of the form, $E = K w h^2$, where K is a constant for given inclinations of the wall and free surface. It follows, as in the case of water pressure, that the center of pressure on AB will be at one-third its height above A. Compare with Art. 24.

For the case of the surcharged wall of Fig. 19, the center of pressure is higher than the third point going from A to F. To determine it, a construction similar to that of Fig. 19 was applied to a vertical wall, the earth surface sloping at the angle of repose from B to R, then horizontal. Assuming $\varphi = \varphi' = 33^\circ\ 41'$ and calling the vertical height of wall $AB = h$ and the height of surcharge $BC = h'$, Fig. 23, the planes of rupture shown by the broken lines and the thrusts were found for a

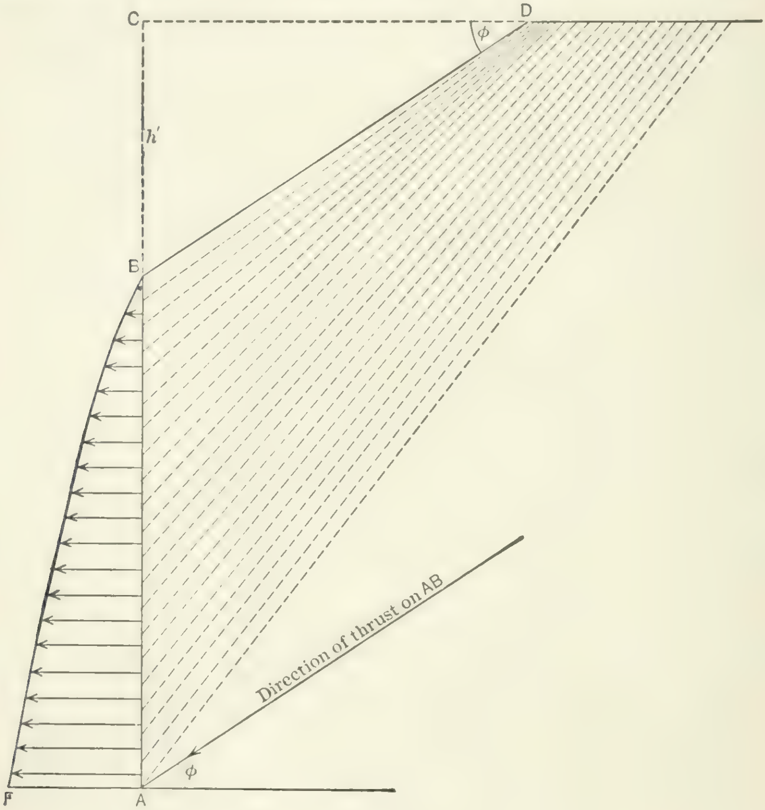

FIG. 23

constant $h' = 10'$ and for $h = 1, 2, \ldots, 20$ ft., in turn. The difference between the successive thrusts gave the pressures on each successive foot (shown by the arrows in magnitude only). Assuming these to act at the center of the divisions, the centers of pressure, for h varying from 5 to 20 ft., were found by taking moments about convenient points. Letting $c =$

ratio of distance from foot of wall considered to center of pressure to height of wall, the results are as follows:

$\frac{h'}{h}$	c	$\frac{h'}{h}$	c	$\frac{h'}{h}$	c	$\frac{h'}{h}$	c
∞	0.333	1.50	0.356	1.00	0.364	....	.....
....	.....	1.25	0.360	0.75	0.364	....	.....
2.00	0.353	1.11	0.362	0.50	0.364	0	0.333

The thrusts were assumed to make the angle $\varphi' = \varphi$ with the horizontal, though here, the arrows giving their magnitudes were laid off horizontally to show the variation better. For other values of φ and φ', the values of c may change somewhat, but they are probably near enough to use in practice for usual values of φ.

44. Surface Sloping Uniformly. Formulas for E. The formula for E is easily derived from Fig. 24, where the letters have the same meaning as in Fig. 20.

Let, φ = angle of repose of earth = JAD,
φ' = angle of friction of earth on wall AB,
i = angle made by free surface with horizontal,
α = angle made by inner face of wall AB with the vertical,
h = vertical height of wall = $AB \cos \alpha$.

As in Fig. 20, we have $ABO = BAy = \varphi + \varphi'$ and from Fig. 19, $NBO = \varphi' + \alpha \therefore NOB = 90° - (\varphi' + \alpha)$. We also have,

$$ADB = \varphi - i,\ ABD = 90° - (\alpha - i),\ BAO = 90° + \alpha - \varphi.$$

In (8), Art. 38, on substituting for $\frac{OI.ID}{AI}$, its maximum value, $\frac{a - \sqrt{ab}}{AD}$, we have the active thrust given in the form,

$$E = \frac{1}{2} w . \overline{BO}^2 \left[\frac{a - \sqrt{ab}}{a - b}\right]^2 \cos OBN \quad . \quad . \quad (13)$$

where, $a = AD$, $b = AO$.

Let $n = \sqrt{\frac{b}{a}} = \sqrt{\frac{AO}{AD}}$ and write the bracketed term as follows:

$$\frac{a - \sqrt{ab}}{a - b} = \frac{1 - \sqrt{\frac{b}{a}}}{1 - \frac{b}{a}} = \frac{1}{1 + \sqrt{\frac{b}{a}}} = \frac{1}{1 + n}.$$

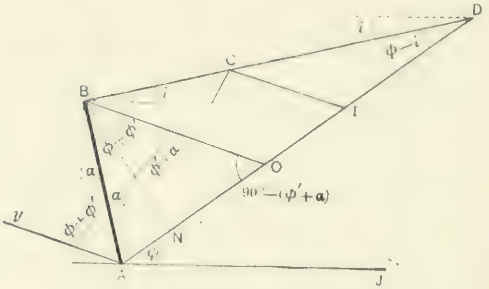

FIG. 24

To find n, from the triangles AOB and ABD, by the law of sines,

$$\frac{AO}{AB} = \frac{\sin(\varphi + \varphi')}{\cos(\varphi' + \alpha)};\quad \frac{AB}{AD} = \frac{\sin(\varphi - i)}{\cos(\alpha - i)}.$$

Multiply these two equations together and extract the square root:

$$\therefore\ n = \sqrt{\frac{AO}{AD}} = \sqrt{\frac{\sin(\varphi + \varphi')\sin(\varphi - i)}{\cos(\varphi' + \alpha)\cos(\alpha - i)}} \quad . \quad (14)$$

From the triangle BOA, by the law of sines,

$$\frac{BO}{AB} = \frac{\cos(\varphi - \alpha)}{\cos(\varphi' + \alpha)}.$$

Substituting in (13), $\frac{1}{1+n}$ for the bracketed term and the value of BO obtained from the last equation, also writing $\cos OBN = \cos(\varphi' + \alpha)$ we have,

$$E = \frac{1}{2}\left[\frac{\cos(\varphi - \alpha)}{n + 1}\right]^2 \frac{w.\ \overline{AB}^2}{\cos(\varphi' + \alpha)}. \quad . \quad . \quad . \quad (15)$$

or since, $AB = \frac{h}{\cos \alpha}$.

$$E = \frac{1}{2}\left[\frac{\cos(\varphi - \alpha)}{(1 + n)\cos\alpha}\right]^2 \frac{w h^2}{\cos(\varphi' + \alpha)} \quad . \quad . \quad . \quad (16)$$

On putting,

$$K = \frac{1}{2}\left[\frac{\cos(\varphi - \alpha)}{(1 + n)\cos\alpha}\right]^2 \frac{1}{\cos(\varphi' + \alpha)} \quad . \quad . \quad . \quad (17)$$

n having the value given by (14), we can write (16),

$$E = K w h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The preceding results all refer to walls with the inner faces battered or those leaning away from the earth.

For overhanging walls or those leaning toward the earth, α being the angle made by the inner face with the vertical, formula (18) again applies provided we put,

$$n = \sqrt{\frac{\sin(\varphi + \varphi')\sin(\varphi - i)}{\cos(\varphi' - \alpha)\cos(\alpha + i)}} \quad . \quad . \quad . \quad . \quad (19)$$

and,

$$K = \frac{1}{2}\left[\frac{\cos(\varphi + \alpha)}{(1 + n)\cos\alpha}\right]^2 \frac{1}{\cos(\varphi' - \alpha)} \quad . \quad . \quad (20)$$

The last two formulas are easily seen to be true by drawing a figure and noting that the new angles involving α can be found from the old by simply changing α to $(-\alpha)$. To avoid mistake, in all the formulas (14)—(20), α is taken positive.

It is important to note that the formulas or graphical methods of this chapter are only applicable to the cases cited in Art. 30 or, usually, when AB lies above the limiting plane, and always for walls leaning towards the earth filling, when α does not exceed about 10°. Also see Art. 62.

45. The case where the free earth surface slopes downward to the rear is sometimes though rarely met with in practice. All the preceding formulas apply directly on simply changing (i) to $(-i)$, as is easily seen on drawing a figure.

NOTE. The formulas above, for all the cases mentioned, are very long and tedious in the applications, except for the special cases to be given in the next article. For all other cases, it will shorten the labor to measure AO and AD on a drawing to a large scale and compute $n = \sqrt{\frac{AO}{AD}}$, by using these values; then if AO and AD have been scaled approximately to four significant figures, K can be computed accurately to three, which generally answers the needs of practice.

46. Earth Level at Top; back of wall vertical. Earth Sloping at angle of Repose. For a horizontal earth surface, back of wall vertical and $\varphi' = \varphi$, (16) takes a simple form. On substituting $\alpha = 0$, $\varphi' = \varphi$, $i = 0$ in (14), we have,

$$n = \sqrt{\frac{\sin 2\varphi \sin \varphi}{\cos \varphi}} = \sin \varphi \sqrt{2}$$

whence by (16)

$$E = \frac{1}{2} \frac{w h^2 \cos \varphi}{(1 + \sin \varphi \sqrt{2})^2} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If in place of taking $\varphi' = \varphi$, we assume $\varphi' = 0$, which may either refer to a perfectly *smooth* wall or to the *horizontal* thrust on a vertical plane in an indefinitely great mass of earth with level surface, the formula for E is still further simplified. Now we have, $\alpha = 0$, $i = 0$, $\varphi' = 0$, $n = \sin \varphi$.

$$\therefore E = \frac{1 - \sin \varphi}{1 + \sin \varphi} \frac{w h^2}{2} = \frac{1}{2} w h^2 \tan^2\left(45^\circ - \frac{\varphi}{2}\right) \quad . \quad (22)$$

The equality of the two trigonometric terms can be verified by putting $(90 - \varphi)$ for x in the known formula,

$$\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}.$$

Eq. (22) was deduced in another way in Art. 41, where it was further shown that, for this case, the plane of rupture bisected the angle between the vertical and the line of natural slope.

Earth surface at angle of repose. For this case, $i = \varphi$ and for a vertical wall $\alpha = 0$, $n = 0$; whence (16) reduces to,

$$E = \frac{1}{2} w\, h^2 \frac{\cos^2 \varphi}{\cos \varphi'}$$

and E makes the angle φ' with the horizontal. If we take $\varphi' = \varphi$, the thrust E acts parallel to the sloping surface and we derive again the formula of Art. 40,

$$E = \frac{1}{2} w\, h^2 \cos \varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

47. Pressure of Fluids. As φ, φ' and i approach zero, the granular mass approaches the state of a perfect fluid, so that at the limit, when $\varphi = \varphi' = i = 0$, we have from (14), $n = 0$ and from (15),

$$E = \frac{1}{2} w.\, AB^2 \cos \alpha = \frac{h}{2} w.\, AB \quad . \quad . \quad . \qquad (24)$$

or the surface pressed, multiplied by w times the vertical distance from the level surface to the center of gravity of the surface, a well known formula for the normal pressure of a perfect fluid weighing w lb. per cu. ft.

It is interesting to note that, when $\alpha = \varphi' = i = 0$, the plane of rupture approaches an inclination of 45° to the horizontal as φ approaches zero, Art. 41.

48. Rankine's formula for the thrust, acting parallel to the sloping surface, on a vertical plane in a granular mass of indefinite extent and not subjected to any external force but its own weight. In Art. 26, it was shown that in such a mass the thrust on the vertical plane could only act parallel to the surface; hence putting $\alpha = 0$, $\varphi = i$ in (14) and (16), we have,

$$\begin{aligned} n &= \frac{1}{\cos i} \sqrt{\sin (\varphi + i) \sin (\varphi - i)} \\ &= \frac{1}{\cos i} \sqrt{\sin^2 \varphi \cos^2 i - \cos^2 \varphi \sin^2 i} \\ &= \frac{1}{\cos i} \sqrt{\cos^2 i - \cos^2 \varphi}. \end{aligned}$$

$$E = \frac{w\,h^2}{2\cos i}\left(\frac{\cos\varphi}{1+n}\right)^2$$

$$= \frac{w\,h^2}{2}\;\frac{\cos^2\varphi\cos i}{\left(\cos i + \sqrt{\cos^2 i - \cos^2\varphi}\right)^2}.$$

Substituting,

$$\cos^2\varphi = \left(\cos i + \sqrt{\cos^2 i - \cos^2\varphi}\right) \times \left(\cos i - \sqrt{\cos^2 i - \cos^2\varphi}\right)$$

and striking out the common factor, we obtain,

$$E = \frac{w\,h^2}{2}\cos i\,\frac{\cos i - \sqrt{\cos^2 i - \cos^2\varphi}}{\cos i + \sqrt{\cos^2 i - \cos^2\varphi}} \quad . \quad . \quad . \quad (25)$$

Rankine's well known formula.

This formula will be proved in Art. 58 from certain general considerations affecting stress, not assuming a plane of rupture extending to the surface. It is recognized as exact by all authorities.

The assumption of a plane surface of rupture for the unlimited mass is then justified, for it leads to (25), an exact formula. The graphical construction of Art. 26, Fig. 10, is thus correct, but that of Art. 21, Fig. 9, or of Art. 32, Fig. 11, must be regarded as approximate. Similarly, the construction, Art. 40, Fig. 20, will give the exact thrust on a vertical plane in an unlimited mass, when AB is taken vertical and φ' is replaced by i. The construction for all other cases is approximate.

It may be remarked that the two special formulas, (22) and (23), may both be deduced from (25) by putting $i = 0$, $i = \varphi$, respectively.

49. Unit Pressure. On putting,

$$K = \frac{1}{2}\cos i\,\frac{\cos i - \sqrt{\cos^2 i - \cos^2\varphi}}{\cos i + \sqrt{\cos^2 i - \cos^2\varphi}}$$

we can write (25) in the form,

$$E = K\,w\,h^2.$$

Regarding h as the variable depth from the surface down to a point on the vertical plane considered, let $(h + \Delta h)$ represent a depth greater by Δh, corresponding to a thrust $(E + \Delta E)$, where ΔE represents the increase in the thrust.

$$\therefore\ E + \Delta E = K w (h + \Delta h)^2$$

On subtracting the first equation from the second and dividing by Δh,

$$\frac{\Delta E}{\Delta h} = K w (2 h + \Delta h).$$

Now regarding Δh, which equals $\Delta h \times 1$, as the area on which ΔE acts, the left member gives the average pressure per sq. ft. over the area $\Delta h \times 1$ and the limit of this, as Δh tends indefinitely towards zero, is defined to be, *the intensity of the pressure at the depth* x. Hence, in the notation of the calculus,

$$\text{Intensity at depth } h = \frac{dE}{dh} = 2 K w h \quad . \quad . \quad (26)$$

The intensity of pressure per sq. ft. thus varies directly as the depth and can be represented by the ordinates of a triangle with one vertex at the surface on a vertical side.

Putting $p = wh$ = vertical pressure in lb. per sq. ft. at depth h, we also have, intensity $= 2 K p$.

Similarly putting $l = AB$ in (15), referring say, to Fig. 20 or Fig. 22 and regarding l as variable, the intensity of pressure per square unit of the face AB, at the point l, is $\frac{dE}{dl}$.

50. Center of Pressure for Uniformly Sloping Surface. The normal component of the thrust on the plane AB, whether it leans forward or backward, is found by multiplying (15) by $\cos \varphi'$ and it can be expressed by the simple relation,

$$E_1 = c\, l^2$$

c being a constant for given φ, φ', α and i. Whence the thrust on the area $dl \times 1$, at the variable distance l from B, measured along BA, dl being small, is nearly,

$$dE_1 = 2\,cl\,.\,dl,$$

and its moment about the point B, Fig. 20, is, $2\,cl^2\,.\,dl$; hence the distance from B, measured along BA, to the center of pressure, is equal to the limit of the sum of such moments, divided by the total pressure or,

$$\frac{\int_0^l 2\,cl^2\,.\,dl}{cl^2} = \frac{2}{3}\,l\,;$$

hence the center of pressure on AB is at one-third its height above A.

51. Surcharge Load, Uniformly Distributed. In Fig. 25 is shown a wall $ABCD$ backed by earth ABP with a level surface BP, on which a uniformly distributed load rests. This load may consist of buildings or locomotives resting on railroad tracks parallel to the wall.

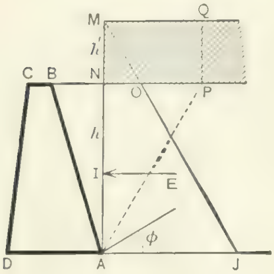

FIG. 25

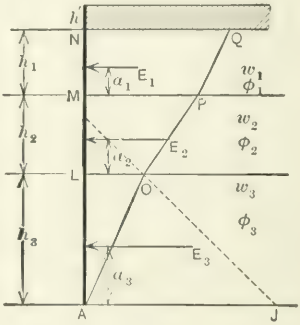

FIG. 26

Let W = weight of this load in lb. per sq. ft. of NP and
h' = height of earth, weighing w lb. per cu. ft. of the same weight as the load.

$\therefore\ w\,h' = W$, so that $h' = MN$ can be computed.

It represents the height of earth that will be supposed to replace the load, since its weight is the same.

In the case of such surcharges, it is a common and commendable practice to assume the thrust on the vertical plane AN as acting horizontally and to compute its amount by aid of (22) by taking the difference of the thrusts on AM and MN, or to proceed by first computing the unit thrusts at A and N by use of (26). By the last method, the unit thrust in the filling at N, which is caused by the load, $p = wh'$, is $2\,K\,wh'$. Lay off, to scale, NO equal to this unit thrust. Similarly, lay off $AJ = 2\,K\,w.\,AM$. The straight line JO should pass through M. The total thrust on AN is the mean of the thrusts at A and N multiplied by the area $AN \times 1$.

By the first method, letting $AN = h$, $NM = h'$, the total horizontal thrust on AN is,

$$E = K\,w\,[\,(h + h')^2 - h'^2] = K\,w\,h\,(h + 2\,h') \quad . \quad (27)$$

In this, as well as in the preceding method,

$$K = \frac{1}{2}\,tan^2\left(45^\circ - \frac{\varphi}{2}\right).$$

Differentiating the expression for E,

$$dE = 2\,K\,w\,(h + h')\,dh,$$

and the distance of the center of pressure I on AN, from N downwards is,

$$\frac{2\int_0^h (h + h')\,h\,d\,h}{h\,(h + 2\,h')} = \frac{2}{3}\,\frac{2\,h + 3\,h'}{h + 2\,h'};$$

and from A upwards,

$$AI = h - \frac{2\,h^2 + 3\,hh'}{3\,h + 6\,h'} = \frac{h + 3\,h'}{h + 2\,h'}\cdot\frac{h}{3}$$

$$= \left(1 + \frac{h'}{h + 2\,h'}\right)\frac{h}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This thrust E acting horizontally to the left at I, is combined with the weight of filling ABN and the weight of wall. to find the resultant on its base. Even if the surcharge extends to *B*, it is wise to ignore the portion over *BN* since the surcharge load is never accurately known, even for warehouses, and train loads are continually increasing. On that account this solution is advised in place of the wall-friction method. There are many applications of it in the chapter on concrete walls.

For future reference, it may be well to note that, $E \times AI = \frac{1}{3} K w h^2 (h + 3 h')$; also the plane of rupture *AF* bisects the angle between *AN* and the natural slope, Art. 41. Where the filling is of the type shown in Fig. 11, Art. 32, with a load on the level portion, it can be reduced to an equivalent weight of earth and the thrust found by the construction of Art. 32. Its point of application on the wall is unknown, but it may be taken at 0.4 the height of *AB* ordinarily.

52. Dock Walls. If the water rises to certain heights above the base, Fig. 25 can be taken to represent a sea wall or a dock wall. Let *AN* in Fig. 26 represent the vertical plane through *A* of the preceding figure and suppose the tide at high water reaches the level *MP* and at low water a lower level. If the water, at high tide, reaches the same height on the front and back faces of the retaining wall, the water pressures on those faces will balance, but if there is a probable lag of the tide back of the wall, its dimensions may have to be arbitrarily increased. In Fig. 26, the earth will be regarded as dry above the level *MP* and saturated below. Let $h_1 = NM$, $h_2 = ML$ and $h_3 = LA$ and call the horizontal earth thrusts on *NM*, *ML* and *LA*, E_1, E_2, E_3, respectively. Also let (w_1, φ_1), (w_2, φ_2), (w_3, φ_3), represent the weights per cu. ft. and the angles of repose, of the earth above *MP*, between *LO* and *MP* and below *LO* respectively.

As before, $h' w_1 = W$ = load on *NQ* in pounds per square foot of *NQ* and from (27),

$$E_1 = \frac{1}{2} \tan^2 \left(45 - \frac{\varphi_1}{2}\right) [w_1 h_1^2 + 2 h_1 W],$$

and from (28), E_1 acts above *MP* the distance,

$$a_1 = \left(1 + \frac{h'}{h_1 + 2\,h'}\right)\frac{h_1}{3}.$$

The uniform load per sq. ft., on the plane MP is now, $W' = W + w_1h_1 = w_2h''$, where h'' is the height of an equivalent load of earth whose weight in lb. per cu. ft. is w_2 or that of the middle stratum.

By (27) and (28),

$$E_2 = \frac{1}{2}\,tan^2\left(45^\circ - \frac{\varphi_2}{2}\right)[w_2h_2^2 + 2\,h_2\,W']$$

and it acts above L, the distance,

$$a_2 = \left(1 + \frac{h''}{h_2 + 2\,h''}\right)\frac{h_2}{3}.$$

Similarly, we replace the load per sq. ft. on LO, $W' + w_2h_2 = W''$, by an equal weight of earth of weight in lb. per cu. ft., w_3 and height h''', so that, $h'''w_3 = W''$, from which h''' can be computed ∴ by (27) and (28),

$$E_3 = \frac{1}{2}\,tan^2\left(45^\circ - \frac{\varphi_3}{2}\right)[w_3h_3^2 + 2\,h_3\,W'')$$

and E_3 acts above A the distance,

$$a_3 = \left(1 + \frac{h'''}{h_3 + 2\,h'''}\right)\frac{h_3}{3},$$

Having found by these formulas, E_1, E_2, E_3, in position and magnitude, the position of their resultant can be found by taking moments about A. This resultant must now be combined with the weight of earth and wall to the left of the plane AY, to get the final resultant on the base. The weight of the submerged filling, is the weight in water, as found by aid of the table in Art. 9. The weight of masonry above high water will be that in air; below high water, the weight in lb. per cu. ft. must be diminished by 64 for salt water. The position of the vertical through the center of gravity of the combined weights of submerged and unsubmerged earth (to left of AY) and masonry, can be found by taking moments about A or any convenient point.

It is very desirable to secure filling with as large an angle of repose in water as possible. Thus river mud is undesirable, φ varying perhaps from 0° to 15°; for sand and clay, dry, say, $\varphi = 36° 53'$, wet, $\varphi = 18° 26'$; whereas rip-rap or cobblestones will have $\varphi = 42°$ to 45°, dry or submerged. Hence the latter is the most desirable filling of all.

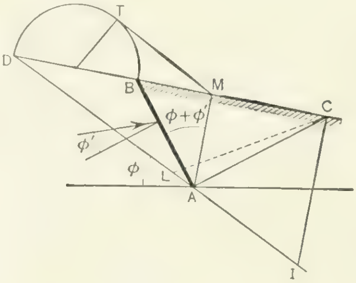

FIG. 27

To illustrate with a numerical example, let rip-rap be placed below JO extended to the wall, JO making an angle of 45° with the horizontal and suppose a sand and clay mixture above OJ to extend to the level NQ. E_2 will be computed for the saturated earth. All lengths given are in feet, weights in pounds. Assume; $h_1 = 5$, $w_1 = 100$, $\varphi_1 = 33° 41'$, for dry earth; $h_2 = 6$, $w_2 = 65$, $\varphi_2 = 18° 30'$, for saturated earth; $h_3 = 10$, $w_3 = 65$, $\varphi_3 = 45°$, for rip-rap in water. Then since the plane of rupture bisects the angle between the vertical and the line of natural slope, Art. 41, the planes of rupture make angles with the vertical of 22° 30′ for rip-rap, 35° 45′ for saturated earth and 28° 10′ for dry earth. AO, OP, PQ, represent these successive planes of rupture, so that $ANQPOA$ is the mass that would slide down if the wall should give. In any computation, regard O as the intersection of the plane of rupture with the outer slope of the rip-rap, so that LO is the horizontal through O, which thus determines h_2.

Assuming the load on NQ, $W = 800$ lb. per sq. ft., we find $h' = W/w_1 = 8'$. Also, since,

$$w_2 h'' = W'' = W' + w_1 h_1 = 800 + 500 = 1300 \therefore h'' = 20',$$
$$w_3 h''' = W'' = W' + w_2 h_2 = 1300 + 390 = 1690 \therefore h''' = 26',$$

simple substitution in the formulas, gives:

$$E_1 = 1500 \text{ lbs.}, \ a_1 = 2.30',$$
$$E_2 = 4640 \text{ lbs.}, \ a_2 = 2.87',$$
$$E_2 = 3470 \text{ lbs.}, \ a_3 = 4.73'.$$

Whence, taking moments about A, the arms of E_1, E_2, E_3, being 18.30′, 12.87′, 4.73′, and dividing the sum of the moments by the sum of the thrusts, E = 9610 lbs., it is found that the resultant E acts 10.8′ above A or *very slightly above the center of AN.* This result follows because of the comparatively small thrust of the rip-rap. If earth was to replace the rip-rap, E_3 would be three times the value given. To justify the computation of E_2, as given, observe, since OP is the plane of rupture for the middle stratum, that the resulting thrust on the vertical plane through O, for the height of this stratum, is transmitted through the intervening earth and rip-rap, to the plane LM and is thus the thrust E_2 on LM.

A similar computation to the above can now be made for low water, the resultant on the base of the wall ascertained and the result compared with the preceding.

It is advisable, when practicable, to tilt the base of sea walls; otherwise, the probable small resistance to sliding on the base may call for a very thick wall.

53. Passive Thrust. The assumptions in the case of passive thrust (or resistance) are exactly those of Art. 36, where one solution is given, the angles φ and φ' will change sign and another solution can be effected by an analysis similar to that of Arts. 38-39. The final result alone will be given here.

In Fig. 27, where AB represents the inner face of the wall, BC the earth surface, draw AD to the left of AB, making the angle φ with the horizontal and produce CB to meet AD at D. Lay off $BAM = \varphi + \varphi'$, to the right of AB and from M, on BC, draw a tangent MT to the semi-circle described upon DB as diameter. Lay off $MC = MT$, draw $CI \parallel AM$ to intersection I with DA produced. Lay off on ID, $IL = IC$, then if LC is drawn, the passive thrust, directed as indicated in Art. 36, is given by w times the area of the triangle CIL and AC is the plane of rupture. This represents the resistance of the wedge of rupture ABC to being pushed up the planes AB and AC by a force, equal and directly opposed to this resistance, acting to the right on the wall, as shown. It is very much larger than the active thrust, particularly when AB is nearer the vertical through A or to the right of it, when L is often found above D. It is assumed that the force on the wall acting to right is applied at ⅓ its height above A; otherwise the earth resistance will not increase from zero at B uniformly downwards and a strict solution is not given by the construction.

54. Ellipse of Stress. It is a well known principle, that for every condition of stress parallel to one plane, there are two planes at right angles to each other on which the stresses are wholly normal.[*] Such stresses are called the *principal stresses* and their directions the *principal axes of stress.*

In Fig. 28, let AB and BC be the planes at right angles to each other on which the stresses are wholly normal. The figure is supposed to represent a cross-section ABC of a right prism, of length unity at right angles to the plane of the paper, with the unit stresses, a, b and r, acting on the three faces AB, BC and CA, the stresses all acting parallel to the plane of the paper. The lengths AB, BC, CA, will be taken as infinitesimal, so that the unit stress on each face can be regarded as uniformly distributed or of uniform intensity over that face.

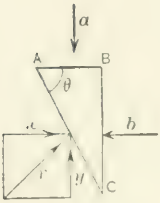

FIG. 28

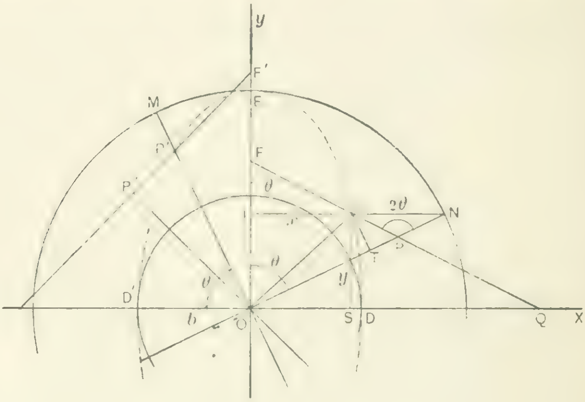

FIG. 29

Let a = normal unit stress on face AB,

b = normal unit stress on face BC,

r = unit stress on face AC.

x and y, components of r perpendicular to BC and AB respectively,

θ = angle BAC.

Assume a numerically greater than b.

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[*] Rankine's Applied Mechanics, p. 98.

The principal stresses a and b are taken as causing compression on the faces AB, BC. If either one is reversed in direction, its sign will be changed in the formulas below.

The three forces $a.AB$, $b.BC$, $r.AC$, are in equilibrium; hence balancing components parallel to AB and BC respectively,

$$x.\,AC = b.\,BC = b.\,AC \sin\theta$$

$$y.\,AC = a.\,AB = a.\,AC \cos\theta$$

$$\therefore\ x/b = \sin\theta,\ y/a = \cos\theta \quad . \quad . \quad . \quad . \quad (29)$$

$$\therefore\ \frac{x^2}{b^2} + \frac{y^2}{a^2} = \sin^2\theta + \cos^2\theta = 1 \quad . \quad . \quad (30)$$

This is the equation of an ellipse, called the ellipse of stress, Fig. 29, in which the components x, y of r are shown as the ordinates to the ellipse at the point R. We have, $r = \sqrt{x^2 + y^2} = OR$, acting upwards for the case shown, EO equal and parallel to a, DO equal and parallel to b and, if we draw $OM \parallel AC$, then $MOD' = \theta$.

With center O, construct circles with radii, a and b and draw $ON \perp OM$ to meet the outer circle at N. The stress $r = OR$, thus acts on a plane parallel to OM, whose normal is ON.

The *obliquity* of the stress is defined to be the angle RON, that the stress makes with the normal to the plane on which it acts.

It is shown in Analytic Geometry, that if from the point where ON cuts the inner circle a line SR is drawn parallel to OE and from N, a line NI parallel to OD, their intersection R is a point on the ellipse, corresponding to the *eccentric angle POE* which equals $MOD' = \theta$, as marked, the corresponding sides being perpendicular. Any radius vector to the ellipse gives the stress on a plane that can be determined; thus if OR is the stress, draw $RN \parallel OD$ to N, the intersection with the outer circle, then ON is the normal to the plane on which OR acts. Drawing $OM \perp ON$, it follows that the stress OR acts on a plane parallel to OM and that its obliquity is RON.

A useful method of drawing an ellipse is as follows: Suppose the axes ox, oy, Fig. 29, drawn at right angles to each other. Lay off on the straight edge of a sheet of paper, distances $FR =$

b, $RQ = a$, then always keeping the points F and Q on the axes oy, ox, as the line FQ is shifted, the point R will describe an ellipse with semi-axes $a = OE$, $b = OD$.

The proof is easy. Thus calling x, y, the coördinates of R and noting that $\angle IFR = \angle SRQ$,

$$\sin IFR = \frac{x}{b},\ \cos IFR = \cos SRQ = \frac{y}{a}$$

$$\therefore \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

which is the equation of an ellipse, in fact of the ellipse of stress.

Comparing the equations above the last with eqs. 29, we see that

$$\angle IFR = \theta = NOF,$$

hence, P being the intersection of the normal ON with FQ, $OP = PF$; also, since $POQ = 90° - \theta = PQO \therefore OP = PQ = PF$.

The external angle FPN of the triangle POF is equal to the sum of the two interior opposite angles, POF, PFO;

$$\therefore FPN = 2\theta.$$

Some important conclusions can now be drawn.

Since $OP = PQ = PF$, $b = FR$, $a = RQ$,

$$b + RP = a - RP \therefore RP = \frac{1}{2}(a - b);$$

$$OP = PF = b + RP = b + \frac{1}{2}(a - b), = \frac{1}{2}(a + b)$$

$\therefore OP = \frac{1}{2}(a + b)$, $RP = \frac{1}{2}(a - b)$, are constant for any point R of the ellipse.

Hence if the plane OM, on which the stress OR can be supposed to act, rotates about O, the point P on its normal, describes a circle about O with a radius $\frac{1}{2}(a + b)$ and the point R describes a circle about P with a radius $\frac{1}{2}(a - b)$, with the condition that OP and PR shall always be equally inclined to OF or that PF shall be made equal to OP.

Thus if the unit stress on a plane $\perp OP'$ is desired, lay off $OP' = OP = \frac{1}{2}(a+b)$; with P' as a center, describe an arc with radius $P'O$, cutting Oy at F' and on $P'F'$ lay off $P'R' = PR = \frac{1}{2}(a-b)$. Then R' is a point on the ellipse of stress and CR' represents the unit stress on the plane whose normal is OP'.

55. *Normal and tangential components of the unit stress on the face AC*, Fig. 28, which is parallel to OM, Fig. 29. On dropping a perpendicular RT upon ON, Fig. 29, we have the component of OR normal to the plane $OM = OT$ and the component parallel to $OM = TR$.

$$OT = OP - TP = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\cos 2\theta \quad . \; . \quad (31)$$

$$TR = PR \sin OPR = \frac{1}{2}(a-b)\sin 2\theta \quad . \; . \; . \; . \; . \quad (32)$$

Note that θ = angle made by the plane AC, Fig. 28, or its parallel OM, Fig. 29, with the minor axis of the ellipse of stress.

In case the principal stress b is tensile, its direction, as well as that of x, is reversed in Fig. 28. The unit normal stress b must now be considered intrinsically negative (x likewise taking the negative sign from the first equation of Art. 54) and the results of Arts. 54–55, will all be found to apply. The student is advised to re-read these articles in connection with Fig. 30, to verify this statement.

Since b is negative, $RP = \frac{1}{2}(a-b)$, will be greater than $OP = \frac{1}{2}(a+b)$ and the point R will fall on the other side of OY from ON. The unit stress on the plane OM, whose normal is ON, is as before, OR in amount and direction. Verify that its normal and tangential components are correctly given by (31), (32) of Art. 55.

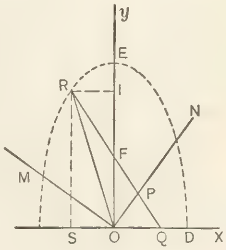

FIG. 30

In the remaining articles of this chapter, the principal stresses a and b will always be considered compressive.

56. The preceding conclusions hold for any state of stress in a solid body and also for a granular mass; but with the condition that the obliquity $R O N$ (or the angle that the stress $O R$ makes with the normal to the plane on which it acts) shall not exceed φ, the angle of friction.

The angle $R O N$, Fig. 29, is a maximum when $P R$ is perpendicular to $O R$, in which case a circle described with P as center and PR radius, will be tangent to $O R$.

$$\therefore\ \sin \varphi = \sin N O R = \frac{P R}{O P} = \frac{a - b}{a + b}$$

For any other position of R (on the circle mentioned) the perpendicular dropped from P upon $O R$ will be less than $P R$ and $\sin N O R$ will thus be less than the value just given. We derive at once from the last formula,

$$\frac{b}{a} = \frac{1 - \sin \varphi}{1 + \sin \varphi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

This gives the ratio of the two principal stresses in the case of earth devoid of cohesion and having at the point of the mass considered the angle of friction φ. This angle is generally assumed constant throughout the mass, as will be done in what follows.

57. In the next figure, 31, the free surface of earth is taken as plane and making the angle i with the horizontal. The infinitesimal wedges of rupture are represented by the triangles $A B C$, having one face $B C$, vertical and one face parallel to the free surface; cases (a) and (b) referring to active thrust where the wedge is about to slide *down* $A C$, cases (c) and (d), to *passive* thrust or resistance, where the wedge is on the point of sliding along the plane $C A$ in the direction $C A$.

The reaction of the earth below $A C$, has a normal component and one parallel to $A C$. The latter or tangential component in cases (a) and (b), is directed up the plane $A C$, but in cases (c) and (d), its direction is from A to C, hence r'', the unit reaction, is drawn as shown for the four cases, always making the angle φ with the normal to the plane of rupture $A C$.*

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[*] It may be remarked that case (d) always corresponds to incipient motion *up* the plane $C A$; but in case (c), a wedge of rupture can be formed when the point A lies below the level of C, the extreme limiting position cor-

The vertical *unit* stress on AB (= weight of vertical prism standing on AB when $AB = 1$) $= r = w\,x \cos i$, where x = vertical depth of AB below the free surface and w = weight per cubic unit of the earth. The *unit* pressure on the vertical plane $B\,C$ is r' and in an unlimited mass of earth, subjected to no external force (as the friction along a wall, etc.) but its own weight, r' acts parallel to AB and to the free surface, Art. 11.

Since either r or r' acts parallel to the plane on which the other acts, they are *conjugate* unit stresses, with a common *obliquity* i, since each makes the angle i with the normal to its plane, Art. 11.

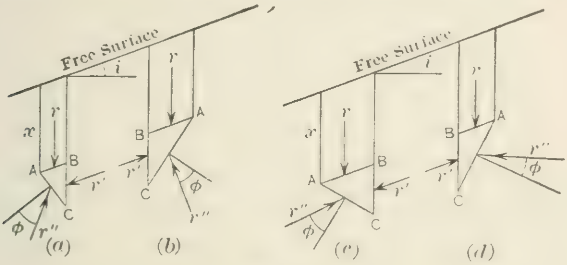

FIG. 31

In cases (a) and (b) of active thrust, the weight of earth over $A\,B$, causes the tendency to slide down $A\,C$, giving an earth thrust opposed in direction to r' as drawn. If the faces $B\,C$ are placed together, the unit thrust of magnitude r' in case (a), must exactly equal and be opposed to the thrust of case (b) or equilibrium will be impossible.

In cases (c) and (d), the unit thrust r', refers to some external force, which is sufficiently great to cause incipient sliding in

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responding to CA parallel to AB. Hence in case (c), according as A is above or below the level of C, the incipient motion is *up* or *down* the plane CA. The student is cautioned that, for any one of the cases, the unit stresses r, r', r'', acting on a wedge of rupture, are not in equilibrium, but that $r.\,AB$, $r'.\,B\,C$, $r''.\,CA$, *do* constitute a system of concurrent forces in equilibrium. Also, it may be noted, on assuming $B\,C$ the same in cases (c) and (d), that when A is below the level of C in case (c), $r.\,AB$ is much larger for this case than for case (d), which explains why r' can be the same for both cases, for the same value of x.

the direction $C\,A$. This means that the wedge $A\ B\ C$ does not slide, but that the slightest increase in r' will cause actual sliding. If we suppose the faces $B\ C$ in cases (*c*) and (*d*) to coincide, the magnitudes of r' for the two cases must be equal, since the thrust for one wedge must equal the reaction of the other.

In all four cases, the three forces $r\,.\,A\ B$, $r'\,.\,B\ C$ and $r''\,.\,A\ C$, must be in equilibrium and the analytical solution is dependent upon this fact. In the ellipse of stress, however, it must be borne in mind that a radius vector as $O\ R$, gives, not the total stress on a plane parallel to $O\ M$, but the *unit* stress on this plane.

58. Conjugate Stresses. In Fig. 29, suppose $r = O\ R$ to represent a unit stress acting on a plane $O\ M$, then its conjugate stress r' must act in the direction $O\ M$. Take $r' = O\ R'$ to represent the magnitude of this stress which acts on a plane OR. Since the stresses are conjugate, they both have the same obliquity $i = N\ O\ R = P'O\ R'$.

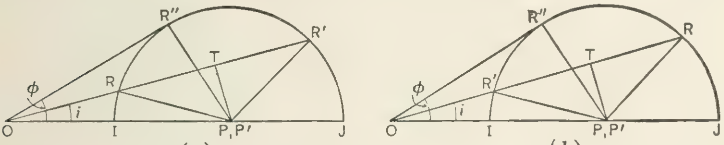
By Art. 54, we have $O\ P = OP'$, $P\ R = P'R'$. To find the ratio of the magnitudes of r' and r:

(1) Let us suppose both $O\ P$ and $O\ P'$ of Fig. 29 to be laid off along the same straight line $O\ J$ from O to P, Fig. 32(*a*) and that lines $O\ R''$, $O\ R$, are drawn making the angles φ and i respectively with $O\ P\ J$. On dropping a perpendicular $P\ R''$ upon $O\ R''$ from P, use P as a center, $P\ R''$ as radius, to describe the semi-circle, which meets $O\ R$ produced at R' and $O\ P\ J$ at I and J. Then $P\ R''$ is equal in length to $P\ R$ of Fig. 29, since for the maximum obliquity φ, $P\ R''$ is perpendicular to $O\ R''$, Art. 56 and $P\ R$ for any point on the ellipse is of constant length. It follows that the triangles $O\ P\ R$ of Figs. 29 and 32(*a*), are equal; also the triangles $O\ P'\ R'$ of both figures are equal, so that the magnitudes of $O\ R$, $O\ R'$ are the same in either figure, though the unit stresses represented by them in Fig. 32 (*a*), are not in true directions. This construction has been made principally as an aid to the better understanding of the next two constructions.

(2) Let us suppose that any convenient length for $O\ P$ in Fig. 32(*a*) is assumed and the remaining construction effected

as before. Evidently the figure will be similar to the former one, the ratio $O R'/O R$ will be the same for either figure, so that the *ratio* of the conjugate stresses can be found from this construction without knowing the magnitude of either one of them.*

(3) Suppose the common obliquity and the magnitude of one of the conjugate stresses, say r, to be known. Draw the lines $O R'$, $O R''$, in Fig. 32(a), making the angles i and φ with $O J$ and lay off to scale on $O R'$, $O R = r$. Since it happens in Fig. 29 that $O R' > O R$, find by trial a point P on $O J$, Fig. 32(a), such that the circle with P as center, $P R$ radius, will be tangent to $O R''$, *that circle being taken which has the greatest radius.* The point P thus found evidently coincides with the P of the first construction and the figure coincides with the preceding throughout. The magnitude of r' is given by the length OR' to scale.



(a) (b)

FIG. 32

As previously found,

$$OP = \frac{1}{2}(a + b),\ PR = P'R' = \frac{1}{2}(a - b) = PI = PJ.$$

$$\therefore\ OJ = a,\ OI = b$$

The ratio r'/r of the conjugate stresses of Fig. 31, is readily computed. Thus if T, Fig. 32(a), is the foot of the perpendicular from P on OR,

$$\frac{OR'}{OR} = \frac{OT + R'T}{OT - RT} = \frac{OT + \sqrt{PR^2 - PT^2}}{OT - \sqrt{PR^2 - PT^2}} =$$

$$\frac{OP\left[\cos i + \sqrt{\sin^2 \varphi - \sin^2 i}\right]}{OP\left[\cos i - \sqrt{\sin^2 \varphi - \sin^2 i}\right]}$$

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* Rankine gives this construction in his "Civil Engineering," p. 320.

$$\therefore \frac{r'}{r} = \frac{\cos i + \sqrt{\cos^2 i - \cos^2 \varphi}}{\cos i - \sqrt{\cos^2 i - \cos^2 \varphi}} \quad . \quad . \quad . \quad (34)$$

This gives the ratio r'/r for cases (*c*) and (*d*), Fig. 31, corresponding to a passive thrust, $r' > r$.

For *active* thrust, cases (*a*) and (*b*), Fig. 31, $r' < r$. Fig. 29 will apply to this case if accents are removed from P', R', and applied to P, R, so that now $O R' < O R$. Fig. 32(*b*) represents the corresponding abbreviated construction, $O R$ being laid off, to some scale $= r$, and a point P on $O J$ found by trial, such that a circle with P as center, $P R$ radius, will be tangent to OR'', *that circle being taken which has the least radius.*

As before, $r = O R =$ vertical unit pressure on a plane parallel to the free surface, at the depth considered, whence $O R'$ to the same scale will represent r'.

The ratio for *active* thrusts is thus, Fig. 32(*b*),

$$\frac{r'}{r} = \frac{OR'}{OR} = \frac{\cos i - \sqrt{\cos^2 i - \cos^2 \varphi}}{\cos i + \sqrt{\cos^2 i - \cos^2 \varphi}}. \quad . \quad . \quad . \quad (35)$$

At the vertical depth x, $r = wx \cos i$; on substituting this value in the last two equations we are conducted to Rankine's formulas for r', the unit pressure on a vertical plane, at a vertical depth x, the direction of the pressure being parallel to the free surface.

The ratio r'/r, for either active or passive thrust, can either be computed from the above formulas or the values OR, OR' can be scaled from the figures and the ratios OR'/OR, computed.

Note, that from one of the similar figures (*a*) and (*b*), Fig. 32, the other can be obtained by simply interchanging R' and R, so that hereafter only one figure need be drawn to find the ratio r'/r.

After the circle of Fig. 32 has been determined as above, we may regard $OR R'$ as no longer making a constant angle i with $O J$, but as having any position from $O J$ to OR''. Then, for any one of these positions of ORR', OR and OR' will represent conjugate stresses with a common obliquity $R O J$. Thus, if $\Delta\, ROP$ in Fig. 29 corresponds to and is thus equal to $\Delta\, R O P$ of Fig. 32(*a*), then as shown above, $\Delta\, R'OP'$ in Fig. 29 is equal to Δ $R'OP'$ in Fig. 32(a); whence stress OR, Fig. 29, acting on plane OR' is conjugate to stress OR' acting on plane OR and both stresses have the common obliquity $ROP = R'OP'$.

When $OR\,R'$, Fig. 32(a), coincides with OJ, the conjugate stresses OR, OR', reduce to the principal stresses OI, OJ. When $OR\,R'$ is revolved to the position OR'', then the conjugate stresses are both equal to OR''. Since the common obliquity is now the maximum φ, the unit stress OR'' is that on either plane of rupture.

59. For *active* thrust, on putting,

$$K = \frac{1}{2} \cos i \cdot \frac{\cos i - \sqrt{\cos^2 i - \cos^2 \varphi}}{\cos i + \sqrt{\cos^2 i - \cos^2 \varphi}},$$

we have from (35), after substituting, $r = w\,x \cos i$,

$$r' = 2\,K\,w\,x \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

which is identical with (26), Art. 49, when $x = h$. It gives the unit stress or intensity of the conjugate stress r' on a vertical plane, at the depth x. Hence the total earth thrust, acting parallel to the surface, on a vertical plane of depth h, is,

$$E = \int_0^h 2\,K\,w\,x\,d\,x = K\,w\,h^2 \quad . \quad . \quad . \quad (37)$$

which is equivalent to (25), Art. 48, proved in an entirely different manner.

The corresponding formulas for passive thrust or resistance are of the same form as (36) and (37), but the value of K is different. Let the student write the value of K.

From (34) and (35), substituting $r = w\,x \cos \varphi$, we have, when $i = \varphi$, for both active and passive thrusts, $r' = w\,x \cos \varphi = r$.

When $i = 0$, the active unit thrust, $r' = w\,x\,\dfrac{1 - \sin \varphi}{1 + \sin \varphi}$,

the passive unit thrust, $r' = w\,x\,\dfrac{1 + \sin \varphi}{1 - \sin \varphi}$.

60. An application will now be made of the ellipse of stress to finding the planes of rupture and the amount and direction of the active pressure on any plane, at a given depth below the free surface of earth, which will be supposed to make an angle i with the horizontal.

In Fig. 33, draw OG perpendicular to the free surface, then the vertical pressure per square unit on a plane OA, parallel to the surface, at a vertical depth x, $= w\,x \cos i = w\,.\,OG = w\,.\,OR$, where OR is vertical and equal to OG. Hence reserving the factor w, the point R will be one point on the ellipse of stress corresponding to $w = 1$. Now the pressure on a plane of rupture through O makes an angle φ with its normal. Suppose this plane revolved around O until its normal takes the position OG and lay off the angle $GOR_o = \varphi$, to the right of OG. Then P and R, having the same meaning as in Fig. 29, remembering that OP and PR are constants, find by trial, the point P on OG, such that an arc of a circle with P as center and PR as radius, will be tangent to OR_o. Drawing $PR_o \perp OR_o$, the obliquity POR_o is a maximum (Art. 56) and equal to φ, the angle of friction.

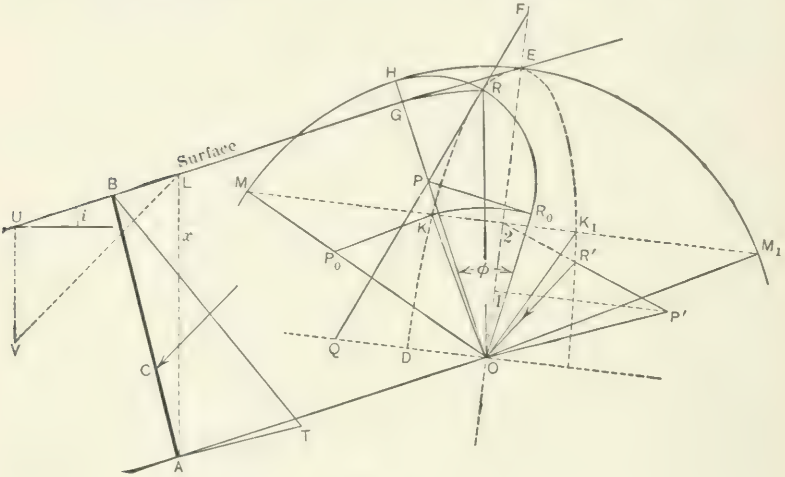

Fig. 33

Lay off $PGH = PR = PR_o$, also extend PR in either direction. Now since R is one point on the ellipse of stress and since by Art. 54, OP and PR must make equal angles with its major axis, lay off $PRF = OP$, whence $\angle POF = \angle PFO$, and draw OF to give the direction of the major axis. On drawing $ODQ \perp OF$, by the method of constructing an ellipse, Art. 54, we have $RF = b$, $RQ = OH = a$. Otherwise, as hitherto proved,

$$OH = OP + PR = \frac{a+b}{2} + \frac{a-b}{2} = a$$

$$RF = PF - PR = \frac{a+b}{2} - \frac{a-b}{2} = b.$$

Next lay off on OF, $OE = a = OH$ and on OQ, $OD = RF = b$. Also mark on the straight edge of a sheet of paper (or other straight edge) the points Q, R, F; then on shifting the straight edge so that the outer points always remain on the axes OQ, OF, produced, the inner point will describe the ellipse of stress.

PLANES OF RUPTURE. One plane of rupture can be found as follows: With center O and radius OR_o describe an arc of a circle, cutting the ellipse at K. Draw $P_oK \perp OK$ and lay off $P_oK = PR_o \therefore \Delta P_oOK = \Delta POR_o$ and $\angle P_oOK = \angle POR_o = \varphi$. Thus it appears that ΔPOR_o is the revolved position of ΔP_oOK, which is in true position, OP_o being the normal to the plane of rupture on which the unit stress is $w \,.\, OK$, this stress making the angle $P_oOK = \varphi$ with the normal to the plane.

An alternative construction is available. Draw a perpendicular through K to the major axis OE to meet a circle described with O as center and $OE = a$ as radius, at M and M_1, then MO coincides in position with P_oO, Art. 54.

If $M\,K\,M_1$ meets the ellipse to the right of OE at K_1, then from symmetry, $\angle K_1OM_1 = KOM = \varphi$; whence OM_1 is the normal to the second plane of rupture on which the unit stress is $w \,.\, OK_1 = w \,.\, OK$.

From the figure, it may be stated as a proposition, that the planes of rupture make equal angles with and on either side of, the major axis, for the case of active thrust here considered.

To avoid confusing the figure, the planes of rupture are not drawn. One is perpendicular to OM, the other to OM_1 nearly coinciding with OK_1 and OK, respectively for the i and φ assumed. This construction for planes of rupture is not so accurate as that given hitherto or the one to follow, Art. 65, since the ellipse is not generally so accurately drawn that the point K is fixed with certainty.

To find the unit stress at O on a plane parallel to AB, draw $OP' \perp AB$ and make $OP' = OP$. Let 1 be the foot of a perpendicular from P' on OE. Lay off along OE, $\overline{12} = O1$, draw $P'2$ and on it lay off $P'R' = PR$. It follows from Art. 54 that $w \,.\, R'O$ is the unit stress at O on a plane parallel to AB,

$R'O$ being the direction of the stress; since for any point R' on the ellipse we must have, $OP' = OP$, $P'R' = PR$, OP' and $P'R'$ making equal angles with the major axis.*

The total thrust on the plane AB in an unlimited mass of earth is found as follows: The thrust at $A = w \cdot R'O$ and it acts $\parallel R'O$. Since the thrust varies uniformly with the depth from the surface, if we lay off $AT \perp AB$, of length $AT = OR'$ and draw BT, $w \times$ area triangle ABT will give the total thrust on the plane AB. Its resultant acts parallel to $R'O$ at C, where $AC = \frac{1}{3} AB$.

When AB is the inner face of a retaining wall and AB makes a greater angle with the vertical than the plane of rupture $\perp OM_1$ (OK very nearly), this construction applies to finding the thrust on the wall, provided $\varphi' \geqq \varphi$. As drawn, AB is less inclined than the normal to OM_1 to the vertical; hence the construction of Arts. 21 or 39 must be used. As walls are generally rough or stepped on the inner face, it rarely happens that $\varphi' < \varphi$. It sometimes occurs by design in the case of experimental walls. In this case if the construction of Fig. 33 gives $R'O$, making a greater angle than φ' with the normal to the wall AB, then the construction of Art. 21 or Art. 39 applies.

The stress on the plane AB, Fig. 33, can be found also by a method, which avoids drawing the ellipse of stress. Thus, let a vertical plane through A intersect the free surface at L and denote AL by x. Then the unit stress r' at A, acting parallel to the surface, is given by the construction Fig. 32(b). Thus when $OR = wx \cos i$, the construction gives, $r' = OR'$. The total stress on the vertical plane AL, Fig. 33, is thus, $\frac{1}{2} r'x$, acting $\frac{1}{3} x$ above A and parallel to the surface. Lay this off to scale, along the surface, say equal to LU and from U draw a vertical line UV of the length, to the same scale, equal to the weight of earth ABL. The resultant of these two forces, LV, gives the amount and the direction of the stress on AB. It acts at the point C.

61. *When the earth surface makes the angle φ with the horizontal*, the construction is simplified. OR being vertical and equal to OG, the angle GOR, Fig. 34, is now equal to φ and the line OR_o coincides with OR. The point P is found at the intersection of a perpendicular at R to OR with OG produced, for then the angle POR is the maximum φ. As before, we lay off $PRF = PO$ to locate a point F on the major axis. Also, $b = RF$, $a = OP + PR$; whence the ellipse of stress can be drawn if desired.

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* Compare with Howe's Retaining Walls, p. 6.

Since RO makes the angle φ with the normal to AO, AO is one plane of rupture. Also, since the conjugate stress to RO acts on a vertical plane in the direction AO (see Fig. 31), making an angle φ with the normal to the vertical plane, the latter is the second plane of rupture, or "limiting plane" of Art. 27. Hence, when $\varphi' \geq \varphi$, to find the unit stress at A on the inner face AB of a battered retaining wall, draw $OP' = OP$ and $\perp AB$. With P' as a center and $P'O$ radius, describe an arc cutting OF produced at 2. On $P'2$ lay off $P'R' = PR$, whence $w \cdot R'O$ will give the unit stress required at A. Next lay off $AT \perp AB$, $AT = OR'$; then w. area triangle ABT will give the total stress on AB. Its resultant acts at C ($AC = \frac{1}{3} AB$), in a direction parallel to $R'O$. As an exercise, take the vertical height of $AB = 1$ foot, to a scale of 5 or 10 inches to 1 foot and assume $w = 1$; then the thrust on $AB = \frac{1}{2} AB . OR' = E = K$. Find the value of K in the following table, corresponding to

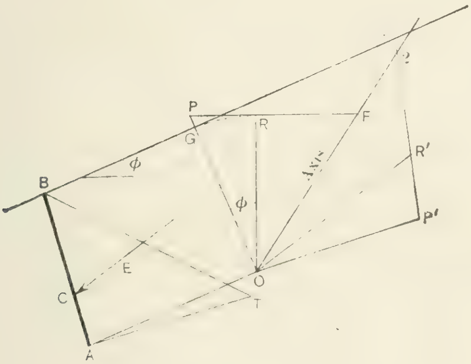

FIG. 34

$\varphi = 33^\circ\ 41'$ and a batter of AB of 2 inches to the foot; also the value of λ = angle made by E with the normal to AB.

In this table,

α = angle made by AB with the vertical,

$K_1 = K \cos \lambda$ = component of K normal to AB.

The values of K_1 given, will be used in the next chapter on the designing of walls. By the graphical method, the values of K can only be counted on to three significant figures, with a possible error of 1 or 2 in the third place and the error in λ may reach 15′.

Batter of AB	α	λ	K	K_1
0″ to 1′	0° 00′	33° 41′	0.4161	0.3462
1″ to 1′	4 46	33 11	.4922	.4119
2″ to 1′	9 28	31 58	.5716	.4849
3″ to 1′	14 02	30 15	.6595	.5697
4″ to 1′	18 26	28 14	.7543	.6646
5″ to 1′	22 37	26 03	.8566	.7696

The values of λ and K were computed from the following formulas,* where ϵ is the angle made by E with the vertical.

$$\tan\left[\epsilon - \left(45^\circ - \frac{\varphi}{2}\right)\right] = \tan\left(45^\circ + \frac{\varphi}{2} - \alpha\right) \tan^2\left(45^\circ - \frac{\varphi}{2}\right) \quad . \quad . \quad (38)$$

ϵ is found from this formula, then $\lambda = 90^\circ - (\epsilon + \alpha)$.

$$K = \frac{1}{2} \frac{\cos(\varphi - \alpha)\tan\alpha}{\cos(\varphi + \epsilon)\cos\alpha} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Remark. In Art. 58, the relation between the vertical unit pressure and its conjugate unit pressure parallel to the steepest declivity, was found. There is a third unit pressure perpendicular to the plane of the first two and therefore horizontal. Since it acts normally to a vertical plane, it is a "principal" pressure. In the general case, Fig. 33, this pressure at the point O acts at right angles to the plane of the paper and is thus conjugate to the greatest principal stress $w \times OE$; hence in magnitude it equals $w \times OD$. The ratio of the latter to the former is given by eq. (33) of Art. 56 and the value of this third principal stress, in lb. per sq. ft., is also given by OI to scale, in Fig. 32(*b*), when $OR = wx \cos i$, x being the vertical depth from the surface to point O. The state of stress at O in any direction can be represented by an ellipsoid, having the three principal stresses as semi-axes.

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* The formulas were derived, by first finding the thrust on the vertical plane through A and then combining it with the weight of earth between that plane and AB. The work is too long to give here. The value of λ may be directly derived from (40) of the next article on changing α to $(-\alpha)$ in that equation, but the above method is shorter.

62. Direction of the Pressure on a Wall Leaning Towards the Earth. Leaning retaining walls are rarely built with the inner face making an angle over 10° with the vertical. For inclinations less than 10° say, the old assumption, that the thrust makes the angle φ with the normal to the wall when $\varphi' \geqq \varphi$, or the angle φ' when $\varphi' < \varphi$, can be used in getting an approximate value of the thrust by previous methods.

These assumptions must, of course, hold when the wall is vertical, but when the wall rests on a bank sloping at the angle of repose, the reaction of the earth must be vertical and directed upwards and there can be no horizontal thrust. Between these extremes, the direction of the thrust must pass through all intermediate positions as the wall rotates from one extreme to the other.

No simple solution is available, but Résal,* by a very complicated analysis, reaches the conclusion, that along the under side of a wall leaning towards the earth, the unit pressure r' on a *vertical* plane Fig. 31, makes the angle φ with the horizontal; hence the conjugate vertical unit pressure r acts on a plane inclined at the same angle φ with the horizontal, *whatever the surface inclination may be.* See Art. 33, where Résal's fundamental hypothesis is stated, from which this conclusion follows.

Résal's conclusion will be adopted, since it gives correct results for the vertical wall and the one leaning at the angle of repose, and reasonable results for intermediate positions.

Fig. 35 shows a ready means for finding the angle λ, made by the thrust on a leaning wall whose inner face is AB', with the normal to AB'. The vertical earth pressure at A is unknown, since no vertical prism of earth extends from A to the surface, as in the case of the battered wall. Hence in Fig. 35 lay off any vertical length OR to represent r and proceed as in Fig. 34 to lay off $POR = \varphi$, draw $PR \perp OR$ and lay off $PRF = PO$. The construction now proceeds exactly as before.

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* "Poussée Des Terres," Vol. 1, 1903.

Thus to find the direction of the thrust on AB', draw $OP' \perp AB'$ and make $OP' = OP$; then lay off $P'2 = P'\mathrm{o}$, $P'R' =$

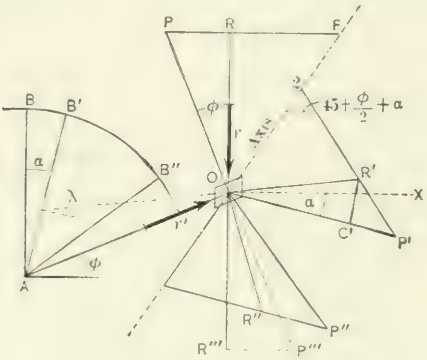

FIG. 35

PR; whence $R'O$ is the direction of the thrust on AB' and it makes the angle $\lambda = P'OR'$ with its normal.

Similarly $R''O$ is the direction of the thrust on AB'' and $R'''O$ (vertical) the direction of the thrust on AO as previously seen. Note that $R''O$ is the direction of the *active* thrust on AB'', corresponding to the wedge of thrust Fig. 31 (*a*) and (*b*), sliding *down* the plane AC.

A formula for *tan* λ can be easily deduced from the figure.

Let α = angle made with the vertical AB by the inner face of the wall.

λ = angle made by thrust on wall with the normal to the wall.

If the component of $R'O$ or $R''O$, etc., parallel to the wall, acts down, λ will be regarded as $+$; when it acts up, λ will be taken as $-$.

To fix the ideas, take AB' as the inner face of the wall.

On drawing OX horizontal, we have, $XOP' = BAB' = \alpha$. Also, since $PO = PF$ (by construction), $\therefore\ \angle\ POF = \angle\ PFO = \angle\ FOX$ and OF bisects the angle $POX = 90^\circ + \varphi\ \therefore\ FOX = 45^\circ + \frac{\varphi}{2}\ \therefore\ P'OF = (45^\circ + \frac{\varphi}{2} + \alpha) = P'2O$. Whence, $OP'R' = 90^\circ - (\varphi + 2\alpha)$ and on dropping the perpendicular $R'C'$ from R' upon OP' and remembering that $OP' = OP$, $P'R' = PR$, we find,

$$P'R' = PR = OP\ sin\ \varphi$$
$$P'C' = P'R'\ cos\ [90^\circ - (2\alpha + \varphi)] = OP\ sin\ \varphi\ sin\ (2\alpha + \varphi)$$
$$R'C' = P'R'\ sin\ [90^\circ - (2\alpha + \varphi)] = OP\ sin\ \varphi\ cos\ (2\alpha + \varphi)$$
$$\therefore\ tan\ \lambda = tan\ R'OP' = \frac{R'C'}{OC'} = \frac{sin\ \varphi\ cos\ (2\alpha + \varphi)}{1 - sin\ \varphi\ sin\ (2\alpha + \varphi)}\ . \quad (40)$$

as found by Résal in a different way.

Ex. Let the student prove, by considering the triangle $OP''R''$, that the formula applies when λ is negative, noting that $tan\ (-\lambda) = -\ tan\ \lambda$.

The following values have been computed by the writer for various values of φ.

α	$\phi = 30^\circ$	$\phi = 33^\circ 41'$	$\phi = 40^\circ$
	λ	λ	λ
0°	+ 30°	+ 33° 41′	+ 40°
5°	+ 29° 26′	+ 33° 02′	+ 39° 09′
10°	+ 27° 31′	+ 30° 42′	+ 35° 56′
15°	+ 23° 48′	+ 26° 03′	+ 29° 03′
30°	0°	− 4° 34′	− 16° 55′
33° 41′	......	− 13° 18′	....
40°	− 17° 53′	− 24° 21′	− 35° 57′
50°	− 27° 31′	....	− 40°
56° 19′	....	− 33° 41′	....
60°	− 30°	....	....

These values of λ will replace φ' in previous graphical or analytical solutions.

63. Mohr's Circular Diagram of Stress. We shall now recur to Fig. 32 of Art. 58. This diagram is repeated in Fig. 36 with additional lines. For *active* thrust, it is well to recall that the ratio of the conjugate stresses, $r'/r = OR'/OR = OR_1/OR$, if $\angle IOR_1 = \angle IOR'$. If x is the vertical depth to the point considered and h is the length of the perpendicular from this point to the free surface, then $r = wx \cos i = w h$ and in what follows, OR to some scale, will be supposed to represent $r = w h$.

Call p and q the normal and tangential components of r, with respect to the plane, parallel to the free surface, on which it acts, Fig. 36, upper figure; also let p' and q' represent the normal and tangential components of r' with respect to the vertical plane on which it acts $\therefore p = r \cos i$, $q = r \sin i$, $p' = r' \cos i$, $q' = r' \sin i$. In the lower figure, r and r' are not shown in true directions, but since $ROJ = R_1OJ = i$, we have for the magnitudes of the components:

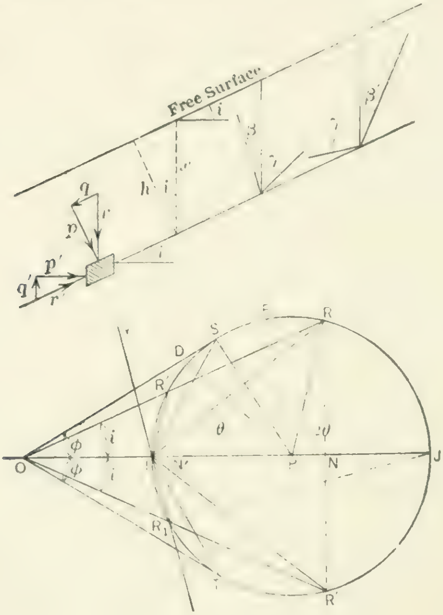

FIG. 36

$$p = ON, q = NR, p' = ON', q' = N'R_1.$$

It has been previously shown that,

$$OI = b, OJ = a \therefore OP = \frac{1}{2}(a + b), \ PR = \frac{1}{2}(a - b).$$

Let $\angle RIJ = \theta \therefore \angle RPJ = 2\theta$,

$$\therefore\ p = OP + PN = \frac{1}{2}(a + b) + \frac{1}{2}(a - b)\cos 2\theta$$

$$q = NR = \frac{1}{2}(a - b)\sin 2\theta.$$

On comparing these two expressions with (31) and (32), Art. 55, and noting the identity, we see that θ is the angle that the plane on which r acts makes with the minor axis of the ellipse of stress. Similarly, letting $\angle R_1IJ = \theta'$.

$$p' = OP - N'P = \frac{1}{2}(a + b) + \frac{1}{2}(a - b)\cos 2\theta'$$

$$q' = R_1N' = \frac{1}{2}(a - b)\sin 2\theta'.$$

Thus θ' corresponds in meaning to θ of eqs. (31), (32), Art. 55 and Fig. 29, and hence it is the angle that the plane on which r' acts makes with the minor axis. Since this is a *vertical* plane, if we rotate the lower figure about I, until R_1I is vertical (or R_1J horizontal), then the planes R_1I, IR, on which r' and r act, will have their true directions; vertical and parallel to the free surface.* The minor axis is parallel to OI in the revolved position and the major axis is perpendicular to it. To get the two planes on which r and r' act in relative position, it was necessary to lay off $r' = OR_1$ and not OR'.

The obliquity of the stress for both planes of rupture is φ; hence IS, in its revolved position, is parallel to one plane of rupture and IT to the other.†

Since R_1 is often near I, in fact, at I when $i = 0$, draw at R_1 accurately a perpendicular to R_1J and mark the upper end V;

---

* This can be verified geometrically. Thus regarding, for the nonce, the circle as having a unit radius, we have in radians $RIR_1 = \frac{1}{2}(RJ + R_1J)$. But this is equal to the obtuse angle between the vertical plane and free surface, for this equals $\left(\frac{\pi}{2} + i\right) = \frac{1}{2}(R_1J + R_1I) + \frac{1}{2}(RJ - R'I) = \frac{1}{2}(RJ + R_1J)$.

† In the general discussion at the end of Art. 58, OR and OR', Figs. 29 and 32, are any two conjugate stresses having the common obliquity $POR = P'OR'$, which can vary from 0 to φ. In Fig. 36, the construction gives, as before, the ratio OR'/OR for any value of ROJ from 0 to φ. When $ROJ = \varphi$, $OR = OR' = OS = OT$ = unit stress on either plane of rupture. The discussion of Art. 63 brings out the additional fact that IS and IT, in their revolved positions, are parallel respectively to the two planes of rupture.

then measure the angles $\beta = R_1IT$, $\gamma = SIV$, made by the planes of rupture with the vertical R_1IV and lay them off in upper Fig. 36, in true position, if desired. These angles can be found very expeditiously and accurately (using a large circle of any convenient size) by this diagram, except when i is near φ, when the intersection R_1 of OR_1 with the arc is at too acute an angle to be reliable. Since the figure can be supposed to refer to any x, the plane of rupture remains at the same inclination and is continuous to the surface.*

When $i = 0$, IV is a vertical tangent to the circle at I and the planes IR, IR_1, IS, IT, have their true directions. Angle $VIS = \frac{1}{2} OPS = \frac{1}{2}(90 - \varphi)$ $\therefore$ the planes of rupture lie on either side of the vertical and bisect the angle between the vertical and the line of natural slope.

The diagram labors under the inconvenience of not giving the conjugate stresses or the planes considered (except for $i = 0$) in true direction, but it is nevertheless a valuable aid in giving at once the magnitudes of the conjugate stresses and the inclinations of the planes of rupture, and it is especially valuable, in a modified form, to coherent earth.

NOTE. After the circle of Fig. 36 is drawn for the given depth x, let it be redrawn with R_1J horizontal. Then at the new position of O erect a perpendicular to the new position of OJ and lay off on it from O a length equal to the semi-major axis $a = OJ$. The semi-minor axis OI, in the revolved position, is now in true direction. On these two semi-axes of lengths $a = OJ$, $b = OI$, the ellipse of stress can be drawn in true position. The ellipse of stress is drawn by a different procedure in Art. 60. After the construction of the ellipse of stress by either method, the stress OR' on a plane AB, Fig. 33, can be found by drawing the triangle $OP'R'$, as explained in Art. 60.

64. The diagram, Fig. 36, refers equally to *passive* thrust, provided we suppose OR', to some scale, to equal, $r = wx \cos i =$

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* The above figure is equivalent to Mohr's "circular diagram of stress" applied to earth pressures. Mohr's demonstration is more general and along different lines to that given here. Prof. O. H. Basquin, in a paper presented, Sept. 9, 1912, before the Western Society of Engineers, gives Mohr's theory in detail, with many interesting applications. Also in the Journal of the Franklin Institute, October, 1882, Mohr's "Graphical Theory of Earth Pressure" is presented by Prof. Geo. F. Swain.

$w\,h$; then $r' =$ passive unit thrust on a vertical plane $= OR = OR''$, Art. 58. It is now easily shown, exactly as in the preceding article, that $R'IJ =$ angle that plane on which $r\ (= OR')$ acts, makes with the minor axis and that $R''IJ =$ angle that plane on which $r'\ (= OR'')$ acts, or the vertical plane makes with the minor axis. Hence we must rotate the lower figure about I until *IR'' is vertical;* then IR' will be parallel to the top slope.

Let $\beta' = R''IT$, $\gamma' = R''IS$; hence as IR'' is now to be considered as vertical, β' and γ' give the angles made by the two planes of rupture with the vertical. In this figure where i is near φ, $\gamma' > 90°$, but still a wedge of rupture is formed, as in Fig. 31(c).

A simpler construction will now be given for finding all the angles, β, γ, β', γ', that the planes of rupture make with the vertical.

65. Planes of Rupture. For measurement of angles, conceive the circle in Fig. 36 to have a unit radius, so that any central angle as SPI is directly measured, in radians, by the arc IS which subtends it. Let D be the mid-point of arc $R'S$ and E the mid-point of arc SR, then by geometry we have,

$$\beta = \angle R_1IT = \frac{1}{2} R_1T = \frac{1}{2} R'S \ \therefore \beta = \angle SPD,$$

$$\gamma = \angle SIV = \frac{1}{2}(IS + IR_1) = \frac{1}{2}(IS + IR') \ \therefore \gamma = \angle DPI;$$

whence, $\beta + \gamma = SPI = 90° - \varphi$.

Also,

$$\beta' = \angle R''IT = \frac{1}{2} TR'' = \frac{1}{2} SR \ \therefore \ \beta' = \angle SPE$$

$$\gamma' = \angle R''IS = \frac{1}{2}(R''J + JS) = \frac{1}{2}(RJ + SJ) = JE.$$

$\therefore \gamma' = \angle EPJ \ \therefore \beta' + \gamma' = 90° + \varphi$, and

$$\beta + \gamma + \beta' + \gamma' = 180°.$$

To make an accurate construction, the scale should be large, and since in similar figures corresponding angles are unaltered, the following construction is suggested. After laying off the angles i and φ from OJ, Fig. 37, take a length of 10 inches in dividers and find by trial a point P on OJ, from which as center, 10 inches radius, a semi-circle can be drawn, as shown touching OS. Next, use the dotted semi-circle (OP diameter) to find very accurately the tangent point S. Then if the line making the angle i with OJ cuts the first semi-circle at R' and R, bisect arcs $R'S$ and SR at D and E, then as proved above, $\beta = SPD$, $\gamma = DPI$, $\beta' = SPE$, $\gamma' = EPJ$. The advantage of the radius 10 inches is that the angles can at once be read off from a table of chords.*

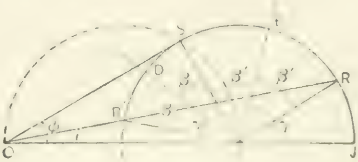

FIG. 37

To derive formulas for β, etc., we shall introduce an auxiliary angle ϵ defined by the relation

$$\sin \epsilon = \frac{\sin i}{\sin \varphi}.$$

In the triangle OPR', we have,

$$\frac{\sin i}{\sin PR'O} = \frac{PR'}{OP} = \frac{PS}{OP} = \sin \varphi \therefore \sin PR'O = \frac{\sin i}{\sin \varphi} = \sin \epsilon.$$

Now, $PR'O = 180^\circ - [i + OPR'] = 180^\circ - [i + (OPS - 2\beta)]$

$\qquad = 180^\circ - [i + (90^\circ - \varphi) - 2\beta]$

$\therefore \sin PR'O = \sin [i + 90^\circ - \varphi - 2\beta] = \sin \epsilon.$

Whence, $i + 90^\circ - \varphi - 2\beta = \epsilon$

$$\therefore \beta = \frac{90^\circ - \varphi}{2} - \frac{\epsilon - i}{2} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

In the same way, by considering the triangle OPR, we derive,

$$\beta' = \frac{90^\circ + \varphi}{2} - \frac{\epsilon + i}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

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[*] Such as is given in Trautwine's Pocket Book.

Having computed ϵ and then β and β' from the formulas above, we find γ and γ' from the relations,

$$\beta + \gamma = 90^\circ - \varphi, \; \beta' + \gamma' = 90^\circ + \varphi.^*$$

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* These formulas are given by Résal in "Poussée des Terres," I, where they are proved by a very laborious method. The construction following, similar to Fig. 37, Résal attributes to M. Maurice d'Ocagne.

CHAPTER IV

DESIGN OF RETAINING WALLS OF STONE OR REINFORCED CONCRETE

66. In the design of retaining walls, it is generally specified that,

(1) *The resultant on the base shall cut it within its middle third,*

(2) *The resultant on the base shall not make an angle with its normal greater than the angle of friction of masonry on earth,*

(3) *The soil pressure at the toes must not exceed the safe bearing value of the earth foundation.*

To these requirements is often added a fourth, introducing a so-called factor of safety:

(4) *The earth thrust on the wall, when multiplied by an assumed factor of safety and combined with the weight of the wall, shall pass through the outer toe of the wall.*

When the earth thrust is computed by the method which includes the whole of the wall friction, the writer recommends that only the normal component of the earth thrust be multiplied by the factor of safety, the friction component remaining unaltered, for reasons given in Arts. 14 and 16. This allows very materially for the occasional decrease of φ' from repeated rains and vibration. For stepped walls, with *no* vibration, possibly both components should be multiplied by the factor of safety. For walls 6 ft. high, a factor of safety of 3.5 may be used; for walls 10 to 20 ft. high, a factor of 3 is recommended, which may be gradually decreased perhaps to 2.5 for walls 50 ft. high and upward, especially those with high surcharges, since the cohesive forces have doubtless come into play before the embankment is finished, thus materially decreasing the earth thrust.

After a thickness of the base has been found to satisfy (4), then it must next be ascertained if (1) is satisfied, by combining the earth thrust (not multiplied by any factor) with the weight of the wall and noting where the resultant cuts the base. If it passes outside its middle third, the thickness must be increased until requirement (1) is satisfied. Then it may be ascertained if (2) and (3) are satisfied, for otherwise the thickness will have to be still further increased. If the thrust is computed by Rankine's formula, then this thrust, acting parallel to the surface, is to be resolved into the two components and each multiplied by the factor of safety. Generally, when Rankine's thrust is used, requirements (1), (2) and (3) are alone specified.

In an exceptional case, when the foundation is rock or very firm and the right of way limited, the resultant on the base may be allowed to pass a little outside the middle third, if requirements (2) and (3) are satisfied. The whole base will not then be in bearing and the leaning of the wall at first will be greater than if (1) is satisfied; but this moving over of the top of the wall should not increase with time if the foundation is rock or firmly cemented gravel.

In connection with requirement (2), see, in addition, the factor of safety against sliding discussed in Art. 16. It is often impracticable, when the base is horizontal, to make this factor as large as 2, but it should be of course greater than 1 and as near 2 as is practicable. Preferably, the base should be inclined.

As to the coefficients of friction and safe soil pressures, see the tables of Art. 8. As it is very desirable to have the resultant on the base, not only cut it within its middle third, but also pass near its center, to better distribute the pressure, a *base slab* or *foundation course* of masonry, projecting beyond the outer and inner faces of the upper part of the wall, is strongly recommended.

Such a base slab should effectually prevent that increased leaning with time, sometimes observed, which is doubtless due to the very imperfect elasticity of the earth, causing a permanent set at each increase of the thrust due to rains or vibration.

67. Tentative Method of Design. In Art. 35, Ex. 5, Fig. 16, a wall is designed by a tentative process, the requirement being that the resultant on the base shall cut it one-third its thickness from the outer toe. If no base slab is used, it will be more satisfactory to have the resultant pass nearer the center of the base the amount to depend upon the state of the soil of the foundation, the more uncertain the foundation, the greater the thickness. For a rock foundation, the thickness computed above is sufficient, but for a soft soil it is inadequate. It is seen then that no hard and fast rule can be laid down in this case, but that after the thickness has been found for minimum requirements, corresponding say to a rock foundation, the engineer must use his best judgment as to increasing the thickness of the base any farther.

The trial and error method illustrated in the example cited is of special value when various batters or thicknesses at top have to be tried to get a satisfactory wall. Here, the top thickness and rear batter are assumed and the front batter found by the construction. If unsatisfactory, another trial is made and so on, until both batters as well as the thickness at top are found to be satisfactory.

In the analytical, direct method to be given in the next article, the two batters are assumed and the thicknesses at top and bottom computed. If the latter are unsatisfactory, other batters must be assumed until finally a satisfactory top thickness is found.

To save labor, the results of computations for various usual types of walls have been given in the following tables.

68. General Formula for Stability of Retaining Walls against Overturning. Let Fig. 38 represent a wall of length unity perpendicular to the plane of the paper and height h, its cross-section being $A\ B\ C\ D$, and let the outer and inner faces and diagonal $D\ B$ make angles with the vertical equal to β, α, and ϖ respectively; also let W denote the weight of the wall and g the horizontal distance from its line of action to the outer toe.

The earth thrust against the face $A\ B$, $K\,w\,h^2$, is supposed to act at one third of its height above the base and to make an angle with the normal to AB equal to λ, the normal component of the thrust being $K_1 w h^2$ (where $K_1 = K \cos \lambda$) and the tangential component, acting down in the direction BA, being $f K_1 w h^2$, where $f = \tan \lambda$.

The angle of friction of earth on wall being φ', when AB *lies above the limiting plane*, Art. 30, the angle λ is generally equal to φ' when $\varphi' \leqq \varphi$, so that $f = \tan \varphi'$, is the coefficient of friction of earth on wall. In the design of walls, since φ' is not generally known in advance, it is usual to put $\varphi' = \varphi$, the angle of friction of earth on earth. For stepped walls, this is always the correct value to use. It is implied here that the thrust will be computed by the method which includes the wall friction.

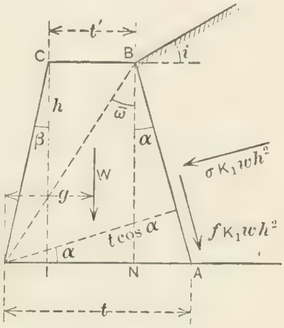

FIG. 38

When AB lies below the limiting plane, the thrust and value of λ must be found after the method outlined in Art. 29 or preferably by the method of Art. 60. When the surface slopes at the angle of repose, or $i = \varphi$, the formulas of Art. 61 are directly applicable and furnish the shortest solution.

When AB leans toward the earth filling and makes a small angle with the vertical—say not exceeding 10°—the thrust is computed as usual for $\varphi' = \varphi$ and $\lambda = \varphi$. In case the wall makes a greater angle with the vertical than 10°, λ may be found from Art. 62.

In the formulas below,

w = weight of earth in pounds per cubic foot,

w' = weight of masonry in pounds per cubic foot,

t = thickness of wall at base in feet,

t' = thickness of wall at top in feet,

σ = factor of safety or number by which the normal component of the thrust is multiplied, the tangential component remaining unaltered, so that the resultant on the base shall pass through the outer toe.

On taking moments about D, we have,

$$Wg + f\, K_1\, w h^2 .\, t \cos\alpha = \sigma\, K_1\, w h^2 \left(\frac{h}{3} \sec\alpha - t \sin\alpha\right),$$

where, $t = h\,(\tan\varpi + \tan\alpha)$.

Dropping the perpendiculars CI, BN, upon AD, we have the moment of the area $ABCD$ about D equal to,

$$\frac{1}{2} CI\,.\,DI \times \frac{2}{3} DI + CI\,.\,IN\left(DI + \frac{1}{2} IN\right) + \frac{1}{2} BN\,.\,NA\left(DN + \frac{1}{3} NA\right)$$

$$= \frac{h^2}{2} \tan\beta\,.\,\frac{2}{3} h \tan\beta + h^2 (\tan\varpi - \tan\beta)\left[h \tan\beta + \frac{h}{2} (\tan\varpi - \tan\beta)\right] + \frac{1}{2} h^2 \tan\alpha \left[h \tan\varpi + \frac{1}{3} h \tan\alpha\right].$$

Whence, on reducing, etc.,

$$Wg = \frac{w' h^3}{6}\left(3 \tan^2\varpi + 3 \tan\varpi \tan\alpha + \tan^2\alpha - \tan^2\beta\right)$$

On substituting this value and $t = h \tan\varpi + h \tan\alpha$ in the first equation and dividing by h^3,

$$\frac{w'}{6}\left(3 \tan^2\varpi + 3 \tan\varpi \tan\alpha + \tan^2\alpha - \tan^2\beta\right) + f\, K_1\, w\, (\tan\varpi \cos\alpha + \sin\alpha)$$
$$= \sigma\, K_1\, w \left(\frac{1}{3} \sec\alpha - \tan\varpi \sin\alpha - \sin\alpha \tan\alpha\right).$$

On simply combining terms, this equation eventually reduces to,

$$\tan^2\varpi + \left[\frac{w}{w'} (f \cos\alpha + \sigma \sin\alpha)\, 2\, K_1 + \tan\alpha\right] \tan\varpi$$
$$= \frac{w}{w'}\, 2\, K_1 \left[\frac{\sigma}{3} \sec\alpha - \tan\alpha\, (f \cos\alpha + \sigma \sin\alpha)\right] - \frac{1}{3}\left(\tan^2\alpha - \tan^2\beta\right) \quad . \quad . \quad . \quad . \quad (41)^*$$

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* Leygue gives this formula in "Annales des Ponts et Chaussées," Nov., 1885.

The formula is adapted to the case where the top of the wall leans toward the earth, or N falls to the right of A, on simply changing $\sin\alpha$ to $(-\sin\alpha)$, $\tan\alpha$ to $(-\tan\alpha)$ and then regarding α as positive.

As this formula is independent of h, it is true for any value of h. If h is given, $\tan\varpi$ is found by solving the quadratic equation (41), whence $t = h(\tan\varpi + \tan\alpha)$ for AB battered and $t = h(\tan\varpi - \tan\alpha)$ for overhanging walls. Also $t' = h(\tan\varpi - \tan\beta)$ for both cases. Should the value of t' ever become negative, the corresponding result must be rejected, since there is then no wall satisfying the assumed conditions.

69. Center of Pressure on Base and Resistance to Sliding. After the value of t has been computed from (41), it is then in order to combine graphically the thrust Kwh^2 on AB (not multiplied by any factor), making the angle λ with the normal to AB, with the weight W of the wall to find the true resultant on the base. Suppose it cuts the base, a feet from its center; then according as $a/t < 1/6$, $= 1/6$ or $> 1/6$, the center of pressure on the base lies in the middle third, at the third point or outside the middle third of the base. In any case, if it should be thought desirable to alter t so that the resultant shall cut the base at any prescribed point, whether the third point or elsewhere, the method used in Ex. 5, Art. 35, will furnish a quick practical solution.

Where the resultant is required to cut the base $\frac{1}{3}t$ from its outer toe, the following formula, deduced in a similar manner to (41), may be used:

$$\tan^2\varpi + \left[\frac{w}{w'}(f\cos\alpha + \sin\alpha)\,4K_1 + \tan\beta\right]\tan\varpi$$
$$= \frac{w}{w'}\,2K_1\,[\sec\alpha - 2\tan\alpha\,(f\cos\alpha + \sin\alpha)]$$
$$+ \tan\beta\,(\tan\beta - \tan\alpha) \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The formula refers to Fig. 38, where N is to the left of A. The formula is adapted to the case where N falls to the right of

A, or for an overhanging wall, on replacing $\sin \alpha$ by $(-\sin \alpha)$, $\tan \alpha$ by $(-\tan \alpha)$.

If the Rankine thrust is to be used in connection with (42), requirements (1), (2), (3) of Art. 66 are alone specified. Values of K_1 and λ have been given in Art. 61 for various batters when $i = \varphi$. For any other case, to find K_1 and λ after the Rankine method, compute the thrust on a vertical plane through A, Fig. 38, extending to the surface, by Rankine's formula, Art. 48, corresponding to $NB = 1$ and $w = 1$. Combine graphically this thrust, acting parallel to the surface, with the weight of earth acting to its left, to find the thrust on AB. Its angle with the normal to $AB = \lambda$ and its component normal to $AB = K_1$. We enter the formula (42) with this value of K_1 and $f = \tan \lambda$.

The angle θ made by the resultant on the base of the wall can be easily found graphically or it can be computed from the formula,

$$\tan \theta = \frac{E \cos (\lambda + \alpha)}{W + E \sin (\lambda + \alpha)}$$

which is easily derived by aid of Fig. 7, Art. 16, on noting that $\tan \theta = \tan NGU = NU/(GH + HU)$. On replacing φ' by λ, we have, $NU = E \cos (\lambda + \alpha)$, $GH = W$, $HU = E \sin (\lambda + \alpha)$. The formula follows. For an overhanging wall, replace α by $(-\alpha)$.

The factor of safety against sliding can be found as in Art. 16.

Finally, the soil pressures can be computed from (3) or (4), Art. 15.

70. Application of (41) to Various Types of Walls. In Fig. 38, since $t/h = (\tan \varpi + \tan \alpha)$, if we assume $h = 1$, the value of $t = DA$ corresponding, equals the ratio of the thickness of the wall at the base to h, for any height of wall. Hence for simplicity, in the following applications of (41) to various types of walls, it will be assumed that $h = 1$. The values of t and t' will be computed for $\varphi = \varphi' = 33° 41'$, corresponding to the slope, 1½ base to 1 rise. The expression for the area of the cross-section $ABCD = A = \frac{1}{2}(t + t')$, likewise represents the volume of masonry for one unit length of wall. Assume $\sigma = 3$, corresponding to usual heights of walls (say 10 to 25 feet) and let t and t' be computed for the two ratios, $\frac{w}{w'} = \frac{5}{6}, \frac{w}{w'} = \frac{5}{8}$, for $i = 0$ and also for $i = \varphi$.

When $i = 0$, the values of K and K_1, corresponding to the case where the wall friction is included, may be taken from Art. 23. These values may be computed from the formulas of Art. 44 or by aid of the graphical method of Art. 21 or that of Art. 39. The same remarks apply to overhanging walls for $i = 0$, but for $i = \varphi$ the formulas of Art. 44 will have to be resorted to.

When $i = \varphi$ and the wall leans away from the earth or is vertical, the Rankine method applies and the values of K and λ will be taken from the table of Art. 61.

71. Type 1. Vertical Rectangular Wall. Fig. 39.

$$\alpha = 0, \ \beta = 0, \ t = \tan \tilde{\omega}.$$

Formula (41) reduces to,

$$t^2 + \frac{w}{w'} 2 K_1 f t = \frac{w}{w'} 2 K_1$$

When $i = 0$, $K_1 = K \cos \varphi' = K \cos \varphi$, is found by aid of (21), Art. 46,

$$K_1 = \frac{1}{2} \frac{\cos^2 \varphi}{(1 + \sqrt{2} \sin \varphi)^2} = 0.109.$$

On substituting in the formula above, $2 K_1 = 0.218$, $f = \tan \varphi = 2/3$, we have,

$$t^2 + \frac{w}{w'} (0.145) t = 0.218 \frac{w}{w'};$$

whence for $w/w' = 5/6$, $t = 0.371$; for $w/w' = 5/8$, $t = 0.326$.

When $i = \varphi$, by (23), Art. 46, $K_1 = \frac{1}{2} \cos^2 \varphi = 0.346$, whence,

$$t^2 + \frac{w}{w'} (0.461) t = 0.692 \frac{w}{w'}.$$

For $w/w' = 5/6$, $t = 0.592$; for $w/w' = 5/8$, $t = 0.530$.

72. Type 2. Vertical Back; Front Face Battered at 2 Inches to the Foot. Fig. 40.

Here, $\alpha = 0$, $\tan\beta = 1/6$ $\therefore$ $\beta = 9^\circ\ 28'$, $t = \tan\varpi$.

$$t^2 + \frac{w}{w'}\,2\,K_1 f t = \frac{w}{w'}\,2\,K_1 + \frac{1}{3}\tan^2\beta.$$

For $i = 0$; as before, $K_1 = 0.109$, $f = 2/3$; whence, for $w/w' = 5/6$, $t = 0.381$; for $w/w' = 5/8$, $t = 0.338$.

When $i = \varphi$, $K_1 = 0.346$, $f = 2/3$, as in Art. **71**, $\therefore$ for $w/w' = 5/6$, $t = 0.597$; for $w/w' = 5/8$, $t = 0.536$.

73. Type 3. Both Faces Battered 2 Inches to the Foot. Fig. 41.

We have, $\tan\alpha = \tan\beta = 1/6$ $\therefore$ $\alpha = \beta = 9^\circ\ 28'$.

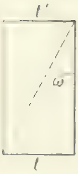

FIG. 39

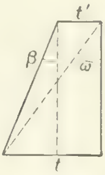

FIG. 40

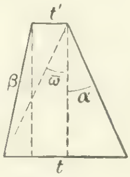

FIG. 41

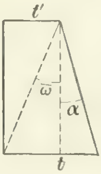

FIG. 42

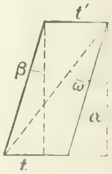

FIG. 43

$t = \tan\varpi + \tan\alpha$, $t' = \tan\varpi - \tan\beta$, $\tan^2\alpha = \tan^2\beta$, $f = 2/3$, $\sigma = 3$; (41) reduces to,

$$\tan^2\varpi + \left[\frac{w}{w'}(1.1511)\ 2\ K_1 + .167\right]\tan\varpi = \frac{w}{w'}\ 2\ K_1 \times 0.822$$

For $i = 0$, by (14) and (17), Art. 44, $n = 0.8434$, $K_1 = K\cos\varphi = 0.143$.

$$\therefore\ \tan^2\varpi + \left[(0.329)\frac{w}{w'} + 0.167\right]\tan\varpi = 0.235\,\frac{w}{w'}.$$

For $\dfrac{w}{w'} = \dfrac{5}{6}$, $\tan\varpi = 0.275$ $\therefore$ $t = 0.442$, $t' = 0.108$.

For $\dfrac{w}{w'} = \dfrac{5}{8}$, $\tan\varpi = 0.240$ $\therefore$ $t = 0.407$, $t' = 0.073$.

For $i = \varphi$, by Art. 61, $K_1 = 0.485$, $\lambda = 31^\circ\ 58'$ $\therefore$ $f = \tan\lambda = 0.624$.

$$\therefore\ \tan^2\varpi + \left[1.076\,\frac{w}{w'} + 0.167\right]\tan\varpi = 0.804\,\frac{w}{w'}.$$

Whence, if $\frac{w}{w'} = \frac{5}{6}$, $tan\ \tilde{\omega} = 0.445 \therefore t = 0.612,\ t' = 0.278.$

If $\frac{w}{w'} = \frac{5}{8}$, $tan\ \tilde{\omega} = 0.404 \therefore t = 0.571,\ t' = 0.237.$

74. Type 4. Front Face Vertical; Rear Face Battered 2 Inches to the Foot. Fig. 42.

$\beta = 0,\ tan\ \alpha = 1/6,\ \sigma = 3.$

For $i = 0$, $K_1 = 0.143$, as in Art. 73 and (41) reduces to the general formula of Art. 73 with $-\frac{tan^2\ \alpha}{3} = -0.009$, added to the right member.

$$\therefore tan^2\ \tilde{\omega} + \left[0.329\frac{w}{w'} + 0.167\right] tan\ \tilde{\omega} = 0.235\frac{w}{w'} - 0.009.$$

$\therefore$ for $\frac{w}{w'} = \frac{5}{6}$, $tan\ \tilde{\omega} = 0.266 = t'.$

$$t = tan\ \tilde{\omega} + tan\ \alpha = 0.266 + 0.167 = 0.433.$$

$\frac{w}{w'} = \frac{5}{8}$, $tan\ \tilde{\omega} = 0.229 = t';\ t = 0.229 + 0.167 = 0.396.$

To ascertain if (1) of Art. 66 is satisfied, combine the true thrust on the wall, $K\,wh^2 = K\,w$, making the angle φ with the inner face, with the weight of the wall, $w'A$, to find the resultant on its base. In this case, $K = 0.172$ and for $\frac{w}{w'} = \frac{5}{6}$, $A = \frac{1}{2}(t + t') = 0.349$. In the construction, it will suffice to put $w = 5$, $w' = 6$, whence the thrust $= 5 \times 0.172 = 0.860$ and the weight of wall $= 6 \times 0.349 = 2.094$. The resultant on the base will be found to pass outside the middle third, not only for $w/w' = 5/6$, but also for $w/w' = 5/8$.

To find the value of t for which the resultant on the base cuts it at a distance $(1/3)\,t$ from the outer toe, use formula (42). This reduces to,

$$tan^2\ \tilde{\omega} + 0.470\frac{w}{w'}\ tan\ \tilde{\omega} = 0.212\frac{w}{w'},$$

on substituting, $K_1 = 0.143$, $\beta = 0$, $f = 2/3$, $tan\ \alpha = 1/6$.

Whence, for $\frac{w}{w'} = \frac{5}{6}$, $\tan \varpi = 0.268 = t'$.

$t = \tan \varpi + \tan \alpha = 0.435$.

For $\frac{w}{w'} = \frac{5}{8}$, $t' = 0.247$, $t = 0.414$.

When $i = \varphi$, it will be found by a construction similar to the above, for the values of t and t' derived below, that the resultant cuts the base within its middle third.

For $i = \varphi$, from Art. 61, $K_1 = 0.485$, $\lambda = 31° 58'$ $\therefore f = \tan \lambda = 0.624$; hence (41) reduces to (see Art. 73),

$$\tan^2 \varpi + \left[1.076 \frac{w}{w'} + .167\right] \tan \varpi = 0.804 \frac{w}{w'} - 0.009.$$

$$\frac{w}{w'} = \frac{5}{6}, \tan \varpi = 0.440 = t', \ t = 0.607;$$

$$\frac{w}{w'} = \frac{5}{8}, \tan \varpi = 0.399 = t', \ t = 0.566.$$

75. Overhanging Wall. Front Face Battered 2 Inches to the Foot; Rear Face Parallel to the Front Face. Fig. 43.

For $i = 0$, $\varphi = \varphi' = 33° 41'$, $\alpha = 9° 28'$, we derive from (19) and (20), Art. 44, $n = 0.7544$, $K = 0.09745$, $K_1 = 0.0811$; also the moment formula is found from (41), by changing $\sin \alpha$ to $(-\sin \alpha)$, $\tan \alpha$ to $(-\tan \alpha)$, to be,

$$\tan^2 \varpi + \left[\frac{w}{w'} (f \cos \alpha - \sigma \sin \alpha)\, 2K_1 - \tan \alpha\right] \tan w$$

$$= \frac{w}{w'} 2K_1 \left[\frac{\sigma}{3} \sec \alpha + \tan \alpha \,(f \cos \alpha - \sigma \sin \alpha)\right]$$

For $K_1 = 0.081$, $f = 2/3$, $\sigma = 3$, $\alpha = \beta = 9° 28'$, this reduces to,

$$\tan^2 \varpi + \left[0.0267 \frac{w}{w'} - 0.167\right] \tan \varpi = 0.169 \frac{w}{w'}.$$

Noting that $t = t' = \tan \varpi - \tan \alpha$, we readily find,

$$\text{for } \frac{w}{w'} = \frac{5}{6}, \tan \varpi = 0.453, \ t = t' = 0.286;$$

$$\text{for } \frac{w}{w'} = \frac{5}{8}, \tan \varpi = 0.409, \ t = t' = 0.242.$$

When $i = \varphi$, by (19) and (20), Art. 44, $n = 0$, $K = 0.2999$, $K_1 = 0.2495$.

Hence the general formula above reduces to,

$$tan^2 \tilde{\omega} + \left[0.083 \frac{w}{w'} - 0.167 \right] tan\ \tilde{\omega} = 0.521 \frac{w}{w'};$$

$$\frac{w}{w'} = \frac{5}{6},\ tan\ \tilde{\omega} = 0.708,\ t = t' = 0.541;$$

$$\frac{w}{w'} = \frac{5}{8},\ tan\ \tilde{\omega} = 0.631,\ t = t' = 0.464.$$

76. Tables of Results for Various Ratios w/w'. The values of t and t' have been similarly made out for $w/w' = 2/3$, $w/w' = 4/5$, and all the results checked graphically by combining the components $3K_1wh^2$, fK_1wh^2, for $h = 1$, to find the earth thrust, which was in turn combined with the weight of wall to find the resultant on its base.

It was found in every instance to pass nearly or exactly through the outer toe. The results were plotted to a large scale and the values of t, taken from the corresponding graph, for $w/w' = 0.60, 0.65, \ldots, 0.85$, recorded in the following tables. The results are correct to within 1 or 2 units in the third decimal figure, three significant figures only having being used in the computations. As illustrated in Art. 74 and explained more generally in Art. 69, the true center of pressure on the base was found and the ratio a/t, computed; also the angle θ made by the resultant on the base with the vertical was ascertained. Only rough average values of a/t and θ are recorded in the tables, as being sufficient for the purpose in view. In some of the cases of Type 5, the resultant was found to cut the base to the right of its center, a/t then being marked negative.

In the table referring to Type 4, certain numbers are placed in parentheses. They refer to the case where the resultant on the base was required to meet it 1/3 its width from the outer toe. The other figures were obtained by using the factor of safety method. It was only for Type 4 that the latter method gave a value of t too small to satisfy requirement (1) of Art. 66.

The value of θ given should not exceed the angle of friction

φ'' of masonry on earth or sliding will occur. The factor of safety against sliding, Art. 16, will be at least *tan* φ''/*tan* θ, and, if possible, this factor should not be less than 2. By reference to the tabular values in Art. 8, it is seen that it is difficult to ensure a factor of 2, especially when $i = \varphi$, so that generally the base should be so inclined that the resultant on it, shall make an angle with its normal much less than the probable angle of friction.

TYPE 1

VERTICAL RECTANGULAR WALL,

$\varphi = 33^\circ 41', \sigma = 3$

Tan i	K	K_1	λ	$\frac{w}{w'}$	$t = t'$	A	$\frac{a}{t}$	θ
0	0.1304	0.1085	φ	0.60	.321	.321	....	....
..	......	......	..	.65	.330	.330	.126	11°
..	......	......	..	.70	.340	.340	....	....
..	......	......	..	.75	.351	.351	....	....
..	......	......	..	.80	.363	.363	.119	12°
..	......	......	..	.85	.374	.374	....	....
2/3	0.4161	0.3462	φ	0.60	.523	.523	....	....
...	......	......	..	.65	.536	.536	.163	18°
...	......	......	..	.70	.552	.552	....	....
...	......	......	..	.75	.568	.568	....	....
...	......	......	..	.80	.583	.583	.161	20°
...	......	......	..	.85	.601	.601	....	....

TYPE 2

REAR FACE VERTICAL; FRONT FACE BATTERED 2 IN. TO 1 FT.

$\varphi = 33^\circ 41', \sigma = 3$

Tan i	K	K_1	λ	$\frac{w}{w'}$	t	t'	A	$\frac{a}{t}$	θ
0	0.1304	0.1085	φ	0.60	.333	.166	.250	....	....
..	......	......	....	.65	.343	.176	.260	.064	13°
..	......	......	....	.70	.353	.186	.270	....	....
..	......	......	....	.75	.363	.196	.280	....	....
..	......	......	....	.80	.374	.207	.290	.051	14°
..	......	......	....	.85	.384	.217	.300	....	....
2/3	0.4161	0.3462	φ	0.60	.529	.362	.445	....	....
...	......	......	....	.65	.543	.376	.460	.051	20°
...	......	......	....	.70	.558	.391	.475	....	....
...	......	......	....	.75	.574	.407	.490	....	....
...	......	......	....	.80	.589	.422	.505	.049	22°
...	......	......	....	.85	.602	.435	.519	....	....

TYPE 3

Front and Rear Faces Battered 2 In. to 1 Ft.

$\varphi = 33° 41', \sigma = 3$

Tan i	K	K_1	λ	$\frac{w}{w'}$	t	t'	A	$\frac{a}{t}$	θ
0	0.1724	0.1435	φ	0.60	.404	.071	.238	....	....
..	......	......	....	.65	.410	.077	.244	.121	14°
..	......	......	....	.70	.418	.085	.251	....	....
..	......	......	....	.75	.427	.094	.260	....	....
..	......	......	....	.80	.437	.104	.270	.119	16°
..	......	......	....	.85	.444	.111	.278	....	....
2/3	0.5716	0.4849	31° 58′	0.60	.566	.233	.400	....	....
..	......	......	....	.65	.576	.243	.410	.084	23°
..	......	......	....	.70	.587	.254	.420	....	....
..	......	......	....	.75	.596	.263	.430	....	....
..	......	......	....	.80	.606	.273	.440	.076	24°
..	......	......	....	.85	.615	.282	.449	....	....

TYPE 4

Front Face Vertical; Rear Face Battered 2 In. to 1 Ft.

$\varphi = 33° 41', \sigma = 3$

Tan i	K	K_1	λ	$\frac{w}{w'}$	t	t'	A	$\frac{a}{t}$	θ
0	0.1724	0.1435	φ	0.60	.391	.224	.307	....	....
..	......	......	....	.65	.400	.233	.316	.175	12°
..	......	......	....	.70	.409	.242	.325	....	....
..	......	......	....	.75	.418	.251	.334	....	....
..	......	......	....	.80	.427	.260	.343	.169	13°
..	......	......	....	.85	.434	.267	.351	....	....
0	0.1724	0.1435	φ	0.60	(.412)	(.245)	(.328)	1/6	....
..	......	......	....	.65	(.416)	(.249)	(.332)	....	....
..	......	......	....	.70	(.420)	(.253)	(.337)	....	....
..	......	......	....	.75	(.426)	(.259)	(.343)	....	....
..	......	......	....	.80	(.432)	(.265)	(.348)	....	....
..	......	......	....	.85	(.437)	(.270)	(.354)	....	....
2/3 .	0.5716	0.4849	31° 58′	0.60	.562	.395	.478	....	....
..	......	......	....	.65	.571	.404	.487	.124	21°
..	......	......	....	.70	.581	.414	.497	....	....
..	......	......	....	.75	.591	.424	.507	....	....
..	......	......	....	.80	.601	.434	.517	.117	23°
..	......	......	....	.85	.610	.443	.576	....	....

TYPE 5

OVERHANGING WALLS, FRONT FACE BATTERED 2 IN. TO 1 FT., REAR FACE PARALLEL TO FRONT FACE

$\varphi = 33^\circ 41'$, $\sigma = 3$

$\tan i$	K	K_1	z	$\frac{w}{w'}$	$t = t'$	A	$\frac{a}{t}$	θ
0	0.0974	0.0811	φ	0.60	.238	.238	....	...
	....		.	.65	.246	.246	−.09	12°
.	. .	. .	. .	.70	.256	.256	....	. .
.	...	.	. .	.75	.267	.267	....	....
.		.		.80	.279	.279	−.07	13°
.		. .	. .	.85	.290	.290	....	....
2/3	0.2999	0.2495	φ	0.60	.453	.453	. .	....
...	.	.		.65	.474	.474	+.013	18°
.	.		. .	.70	.493	.493	. . .	. . .
...	. .	.		.75	.512	.512	....	....
...	.... ..	.		.80	.530	.530	+.019	19°
. .	.... ..	. .	.	.85	.547	.547	....	....

It has been before remarked, Art. 40, that the case $i = \varphi$, or the surface sloping indefinitely at the angle of repose with a corresponding plane of rupture of infinite extent, parallel to the surface, is never realized in practice. The actual case of the surcharged wall is shown in Fig. 11, Art. 32, where the plane of rupture is finite and the thrust is often much less than for the slope of indefinite extent.

The values of t and θ in the tables for $i = \varphi$, are thus in excess for any practical case. The case of the ordinary surcharged wall of one type only will be especially considered in Arts. 79, 80.

No foundation course or slab was assumed in computing the tabular values. Where one is used, the proper treatment will be found in the articles pertaining to concrete walls.

77. Comparison of Results. Applications. A comparison of volumes (column A), for the batters assumed, will show that for $i = 0$ the types are economical in the order,

3, 5, 2, 4, 1,

and for $i = \varphi$ (or $\tan i = 2/3$), in the order,

3, 2, 4, 5, 1.

The values of a/t, for the various types, increase, for $i = 0$, in the order, 5, 2, 3, 1, 4, and for $i = \varphi$, in the order, 5, 2, 3, 4, 1. The variation is very great and it shows that if the walls were designed so that the resultant on the base should cut it one-third its width from the outer toe, the variation of the factor of safety would be very large. It is on that account that the writer prefers the factor of safety method, at least in comparing different types of walls.

In connection with sliding, type 1 shows the smallest value of θ and type 3 the largest value, the variation for all the types not being very marked for either case of surface slope.

In designing walls, the weights w, w', of earth and masonry in lbs. per cu. ft., have to be assumed. If we assume $w = 100$, by use of the values given in Art. 8, we find the ratio w/w', for:

Granite ashlar	0.61
Mortar rubble granite	0.65
Mortar rubble limestone	0.67
Reinforced concrete	0.67
Plain concrete	0.70
Sandstone ashlar	0.71
Dry rubble granite	0.77
Dry rubble limestone	0.80
Common brick masonry	0.80
Dry rubble sandstone	0.91
Cinder concrete	0.91

Dry rubble should only be used for low walls, for which the tabular thickness should be increased 1/4 to 1/3 perhaps.

Recalling, Art. 70, that the tabular t is the ratio of thickness to height for any height of wall and similarly for t', consider a plain concrete wall and assume $w' = 143$, $w = 100$; required the thickness at base and top of a retaining wall 20 ft. high, the thickness at top lying between 1.5 and 3 ft., for $i = 0$.

For Type 2, thickness at base = 0.353 × 20 = 7.06 ft.,
“ “ 2, “ “ top = 0.166 × 20 = 3.33 “
“ “ 3, “ “ base = 0.418 × 20 = 8.36 “
“ “ 3, “ “ top = 0.085 × 20 = 1.70 “

Type 3 appears the most suitable and it will be chosen, especially since it is the cheapest. For other ratios than given in the table, values of t can be interpolated. The application is too obvious to require an example.

78. As limited right of way is often a controlling feature in retaining wall construction. Type 3 is the favorite, especially as it is economical and the form work for concrete is simpler than for the overhanging wall, which is next to it in economy. On that account, the following table is presented for that type, only with the front face battered 1 in. to the foot, the rear face being battered 0, 1, 2, 3, 4, 5 in. to the foot in turn. As before, $\varphi = 33^\circ 41'$ or $\tan \varphi = 2/3$ and $\sigma = 3$.

For the case $i = 0$, $\tan \alpha = 5/12$, the computed value of t' is negative and there is no solution. If we suppose $t' = 0$ with the computed values of t, we have the case of triangular walls corresponding to a safety factor very slightly greater than 3.

It will be observed that A decreases as $\tan \alpha$ increases, but this saving in volume is offset by an increase in θ, particularly when $\tan i = 2/3$ or $i = \varphi$. The angle of friction of masonry on wet clay has been given from 11° to 18°; for sand, dry clay and gravel, 22°, 27° and 31°, respectively. Evidently from the values of θ given in the table for $i = \varphi$, the bases must be tilted upward.

TYPE 3

FRONT FACE BATTERED 1 INCH TO FOOT, REAR FACE WITH BATTERS RANGING FROM 0 TO 5 INCHES TO THE FOOT

$\tan i$	$\tan \alpha$	K	K_1	λ	$\frac{w}{w'}$	t	t'	A	$\frac{a}{t}$	θ
0	0	0.1304	0.109	φ	2/3	.339	.256	.298	.088	12°
...	....	....	....	....	4/5	.366	.283	.325	.082	12.6°
...	1/12	0.1520	0.127	....	2/3	.378	.212	.295	.127	12.6°
...	....	....	....	....	4/5	.402	.236	.319	.117	13.5°
...	2/12	0.1715	0.143	....	2/3	.407	.157	.282	.133	15.5°
...	....	....	....	....	4/5	.430	.180	.305	.135	14.2°
...	3/12	0.1970	0.164	....	2/3	.431	.098	.264	.163	12.8°
...	....	....	....	....	4/5	.450	.117	.283	.167	15.1°
...	4/12	0.2245	0.187	....	2/3	.453	.037	.245	.167	14°
...	....	....	....	....	4/5	.469	.053	.266	.167	15.5°
...	5/12	0.2525	0.210	....	2/3	.460	—	....	....	....
...	....	....	....	....	4/5	.474	—	....	....	....
2/3	0	0.4161	0.3462	33° 41′	2/3	.544	.461	.503	.074	19.3°
...	....	....	....	....	4/5	.561	.478	.520	.084	22°
...	1/12	0.4922	0.4119	33° 11′	2/3	.557	.391	.474	.100	21.2°
...	....	....	....	....	4/5	.583	.417	.500	.094	22.6°
...	2/12	0.5716	0.4849	31° 58′	2/3	.576	.326	.451	.115	22.1°
...	....	....	....	....	4/5	.602	.352	.477	.111	23.7°
...	3/12	0.6595	0.5697	30° 15′	2/3	.583	.250	.417	.138	23.7°
...	....	....	....	....	4/5	.603	.270	.437	.130	25.3°
...	4/12	0.7543	0.6646	28° 14′	2/3	.583	.167	.375	.153	24.8°
...	....	....	....	....	4/5	.598	.182	.390	.164	31°
...	5/12	0.8566	0.7696	26° 03′	2/3	.582	.082	.332	.182	26.1°
...	....	....	....	....	4/5	.593	.093	.343	.184	27.9°
...	5/12	0.8566	0.7696	26° 03′	2/3	(.603)	(.103)	(.353)	(.167)	25.8°
...	....	....	....	....	4/5	(.609)	(.109)	(.359)	(.167)	27.4°

79. Surcharged Walls, Fig. 44; Type 2; Back of Wall Vertical, Exterior Face Battered 2 Inches to the Foot. The earth surface extends from C, at the angle of repose upward to a horizontal surface IL, a vertical distance h' above the top of the wall, whose height is h. Assume $\lambda = \varphi = \varphi'$; then to find K_1, for a given h'/h, as ½, lay off $h = 10$ (say) and $h' = 5$ and by the construction of Fig. 19, Art. 38, for $w = 1$, find $E = K h^2 \therefore K = E/h^2$ and $K_1 = K \cos \varphi$.

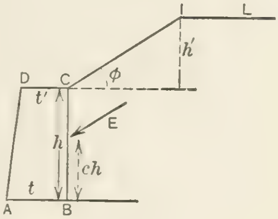

FIG. 44

To find the center of pressure, the values of c of Art. 43 were plotted, the points connected with a curve and the values for various ratios of h'/h taken from the graph, some of them being only roughly approximate. The interpolated values are enclosed in parentheses. The values of t were found by the use of (41) of Art. 68 modified, using the values of c and K_1 as derived above.

SURCHARGED WALLS

$\frac{h}{h^1}$	$\frac{h^1}{h}$	K_1	c	t	
				$w/w' = 2/3$	$w/w' = 4/5$
0.0	∞	.346	.333	.548	.589
0.1	10.00	.313	(.338)	.532	.570
0.2	5.00	.293	(.341)	.522	.559
0.3	3.33	.279	(.344)	.515	.553
0.4	2.50	.270	(.347)	.511	.548
0.5	2.00	.261	.351	.507	.545
0.6	1.67	.253	.354	.504	.541
0.8	1.25	.238	.360	.497	.534
1.0	1.00	.225	.364	.488	.524
1.2	0.83	.214	.364	.479	.516
1.5	0.67	.202	.364	.468	.504
2.0	0.50	.188	.364	.456	.489
3.0	0.33	.166	(.363)	.432	.465
5.0	0.20	.143	(.360)	.408	.439
10.0	0.10	.130	(.355)	.385	.414
∞	0.00	.109	.333	.346	.374

Summary of results for surcharged wall, Fig. 44. $\varphi = \varphi' = 33° 41'$, $\sigma = 3$, $\alpha = 0$, *tan* $\beta = 1/6$, $h = 1$. *The width of base for any* h = (*tabular value of* t) $\times$ h.

The value of θ, or the angle made by the true resultant (for $\sigma = 1$) on the base with the vertical varies from 13° to 14° when $h'/h = 0$, to 20° to 22°, when $h'/h = \infty$.

80. Wall with Surcharge Extending over the Top. Fig. 45. *Type 2; back of wall vertical; exterior face battered 2 inches to the foot.* The earth surface extends from D, upward at the angle of repose, to a horizontal surface IL, a vertical distance h' above the top DC of the wall.

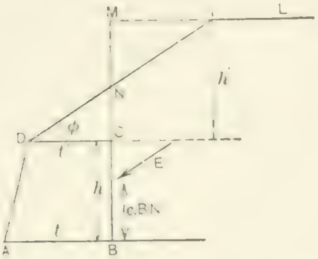

FIG. 45

The values of K_1 and c may be derived from those given in the preceding table, for the ratio NM/BN of Fig. 45 corresponds to h'/h of Fig. 44, in finding K_1 and c. Let $BN = h_o$; then taking moments about A, as in Art. 68, we derive, after reduction,

$$\tan^2 \varpi + f \frac{w}{w'} 2K_1 \left(\frac{h_o}{h}\right)^2 \tan \varpi = \sigma c \frac{w}{w'} 2K_1 \left(\frac{h_o}{h}\right)^3 + \frac{1}{3} \tan^2 \beta,$$

the symbols ϖ, β, σ, w and w', having the meanings given in Art. 68.

The weight of earth DCN over the wall was neglected in finding the resisting moment, which is on the side of safety.

BURIED WALLS

$\frac{h'}{h}$	c		$\frac{h'}{h}$	c	
	$w/w' = 2/3$	$w/w' = 4/5$		$w/w' = 2/3$	$w/w' = 4/5$
0	.346	.374	2.6	.830	.942
0.1	.406	.450	2.8	.838	.950
0.2	.452	.494	3.	.845	.958
0.3	.490	.540	3.5	.860	.974
0.4	.520	.580	4.	.872	.989
0.5	.556	.617	4.5	.882	1.004
0.6	.586	.652	5	.890	1.017
0.7	.614	.680	5.5	.895	1.025
0.8	.636	.710	6	.900	1.032
0.9	.658	.733	6.5	.899	1.038
1.0	.675	.753	7.	.902	1.041
1.2	.708	.792	7.5	.904	1.043
1.4	.735	.830	8.	.906	1.047
1.6	.757	.862	9	.909	1.053
1.8	.776	.890	10	.912	1.055
2.0	.792	.910	15	.927	1.080
2.2	.807	.924	20	.943	1.104
2.4	.820	.934	∞	1.012	1.171

Summary of results for wall with surcharge extending over the top. $\varphi = \varphi' = 33° 41'$, $\sigma = 3$, $\alpha = 0$, *tan* $\beta = 1/6$, $f =$ *tan* $\varphi = 2/3$, $h = 1$. *The width of base for any* $h =$ (*tabular value of* t) $\times h$.

In the computations, a trial t' was assumed, then CN and BN computed. For the ratio, NM/BN (corresponding to h'/h of Fig. 44) the values of K_1 and c were interpolated from the table of Art. 79. Inserting these values in the formula, *tan* $\tilde{\omega} = t$ (for $h = 1$) is computed and then $t' = t - 0.167$. If this value of t' does not agree with the one assumed, a new trial t' must be taken and the work repeated. Generally one or two trials sufficed.

The true resultant on the base makes an angle with the vertical which varied from 13° to 14° for $h'/h = 0$ to 23° to 26° for $h'/h = \infty$. Provision must be made for these large inclinations.

EXAMPLE. *In Fig* 45, *suppose* $h = BC = 20$ *ft.*, $h' = CM = 120$ *ft.* $\therefore h'/h = 6$ $\therefore$ *the width* AB *of wall, for the assumptions above, and* $w/w' = 2/3$ *should be* $20 \times 0.9 = 18$ *ft.*

The table of thickness for the buried walls was made out expressly to show the greatly increased values of t required when the earth is allowed to extend over the top of the wall. If walls have been designed for the previous case and the earth is carelessly dumped over them, failure will almost certainly ensue.

An inspection of the last two tables will show that the values of t vary very much with the height of the surcharge. When h'/h is large, the result approaches that for $i = \varphi$, which can be substituted for it, the solution for $i = \varphi$ being very simple.

When heavy loads, as locomotives, rest on the level portion of the surcharge, the weight must be replaced by an equal weight of earth and the method of Art. 32 applied or when h'/h is large, the solution for $i = \varphi$ may be sufficiently near, remembering that in the latter case, the thrust is applied at 1/3 the height, whereas in the former case the point of application is higher.

In the computations pertaining to the previous tables, it is well to recall that for $i = \varphi$, the Rankine method was used in finding the thrust on other than overhanging walls. It is exact for such cases and furnishes the values of K_1 and λ needed for use in (41). From the examples worked in Art. 35, it is seen to be very inexact when i is small or zero, for walls having little or no batter on the interior face; hence the method including the wall-friction, was used throughout for level-topped filling. However, as noted in Art. 35, the two methods do not differ much in their results when the interior batter is large—say 3 to 5 in. to 1 ft., so that the Rankine method can be used for such batters where desirable. It is fortunate that it is so, for the Rankine method is especially convenient where the filling is subjected to heavy loads and it will be freely used in the following articles on the design of reinforced concrete walls.

Remark. It is important to use a proper material for the filling of retaining walls. A granular material as sand, gravel or rip-rap is good, but as the amount of clay increases, the physical characteristics are more variable, especially at the time of heavy rains, which may materially alter the coefficients of friction and cohesion unless the drainage is efficient.

It is inadvisable to use clay in a nearly pure state as a filling, especially the kind of clays that swell on exposure, sometimes with enormous force.

Clay in bank often stands at a steep slope, even for years, and then gives way. Under the influence of the sun and the weather generally, it has been known to run, somewhat like a viscous substance. In fact its properties are so variable that each clay bank has to be studied by itself. For a foundation course, protected from the atmosphere, it is generally reliable when sufficiently stiff; but when excavated and dumped behind a retaining wall, it is difficult to say what its pressure may become in the course of time.

81. Reinforced Concrete Retaining Walls. In the Appendix will be found a discussion of reinforced concrete beams or slabs, such as are met with in the design of retaining walls, where the faces are often inclined to each other and sometimes, more than one set or layer of rods is used. In the case of the wedge-shaped slabs, a certain assumption was necessary to obtain workable formulas. The results are on the side of safety and this feature should appeal to practical men.

The usual formulas and diagrams for prismatic beams, are obtained as special cases of the general solution. Certain practical data are given, near the end of the Appendix, relating to the best practice of the present day, which should be carefully read. If the reader has already studied the subject of reinforced concrete beams in existing text books, he will be all the better prepared to follow the reasoning and make proper applications of the formulas and methods proposed.

The concrete generally used in retaining wall construction is a mixture, by volume, of one part Portland cement to two parts sand or fine aggregate to four parts coarse aggregate, or as usually abbreviated, a 1 : 2 : 4 concrete.

To save repetition, in all the examples that follow, the ratio of the modulus of elasticity of steel to that of the concrete, will be taken as $n = 15$ and the following working stresses will be used:

Tensile stress in steel	$f_s =$	16,000 lb. sq. in.
Bending stress in concrete	$f_c =$	650 lb. sq in.
Shear as a measure of diagonal tension in a beam without web reinforcement	$v =$	40 lb. sq. in.
Concrete in shear not combined with tension or compression-punching shear		120 lb. sq. in.
Bond stress on plain round or square rods	$u =$	80 lb. sq. in.
Bond stress on plain round or square rods with hooked ends, bent 180° to a radius of 3 diameters and with a short length beyond the bend	$u =$	100 lb. sq. in.
Bond stress on deformed bars, varying with the form	$u =$	80 to 150 lb. sq. in.

To secure sufficient bond resistance, plain bars shall be embedded in the concrete 50 diameters and deformed bars or plain bars with hooked ends (with an assumed $u = 100$ lb. sq. in. for either) 40 diameters.

The soil pressure allowed will be.................. 5,000 lb. sq. ft.

It is often the case that the bearing power of the soil determines the length of base of the wall. The safe bearing capacity of dry sand or clay mixed with sand, is from 4000 to 6000 lb. sq. ft. and for more compacted soils or those with an admixture of gravel, over 8000 lb. sq. ft.; but for soft clay, 2000-4000 and for quicksands, alluvial soils, 1000-2000 lb. sq. ft. only is allowed.

It is very necessary to thoroughly drain the filling, which can be effected by drains placed just back of the wall and on top of the footing and by "weep holes" through the wall at intervals of about 25 feet. The weep holes may be from $3''$ to $4''$ diameter and must extend through the wall, just above the ground surface, where a longitudinal drain should be constructed to carry off the water.

Plain concrete walls should have vertical expansion joints 40 ft. apart at most. The stresses due to shrinkage and temperature changes in reinforced walls, are usually provided for by the steel, in amount, at least ⅓ of 1% of the cross-section of the wall and well distributed near the exposed surface of the concrete.

The coefficient of friction f' of masonry on dry clay is given as 0.5 to 0.6 and on wet clay, as only 0.2 to 0.33, the latter figure referring to moist clay. On dry earth, it is about ½ to ⅔ and on dry sand or gravel ⅔ to ¾.

Sliding is to be feared in light walls and the resistance of the wall to sliding should be the first thing tested. The weight of wall and filling just over it, multiplied by the coefficient of friction, should be greater than the horizontal earth thrust.

The coefficient of friction f' of wall on earth will generally be taken at 0.5 in what follows.

In all the retaining walls that will be examined, the following values are specified:

w = weight of earth per cu. ft. = 100 lbs.,
w' = weight of concrete per cu. ft. = 150 lbs.,

angle of repose of earth $\varphi = 33^\circ\ 41'$, corresponding to the natural slope 2 on 3 and to a coefficient of friction $f = \tan\varphi = \frac{2}{3}$. The surcharge will be taken at 800 lb. sq. ft. on the level surface of the earth, corresponding to an additional load of earth 8 ft. high. The design of the following walls is in reality a partly tentative process, requiring a little preliminary testing of assumed dimensions. To save repetition, only the final design will generally be given.

The width at the top has been taken at 2 feet. Perhaps 1.5 ft. or less would suffice for highways near the wall and something over 2 feet would be more desirable where railway tracks are placed parallel to the wall. In the first example below, railway tracks will be assumed, and since the trains can occupy either or all of the parallel tracks, the surcharge should be taken as extending to a point or points that will subject the wall or the soil to the most trying conditions.

82. Design of a Trapezoidal Wall with Base Slab. In Fig. 46 is shown the first wall that will be examined, the dimensions being marked on the figure. Let us first examine the part *ABCD* above the footing.

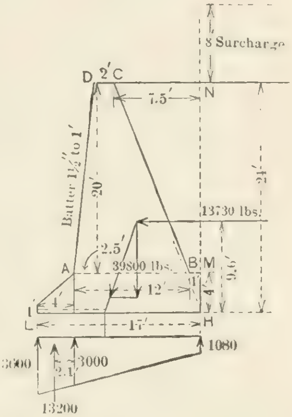

FIG. 46

Conceive the surcharge as extending up to a vertical plane through *B*. The height of this plane $= h = 20'$, height of surcharge $= h' = 8'$; hence by eq. (27), Art. 51, the earth thrust on this plane, acting horizontally to the left, as given by Rankine's formula, for a foot length of wall is,

$$E = \tfrac{1}{2}\, tan^2 \left(45^\circ - \frac{\varphi}{2}\right) w \left[\, (h + h')^2 - h'^2 \right]$$

$$= 0.143 \times 100\ (28^2 - 8^2) = 10{,}296 \text{ lbs.}$$

and it acts above *B*, a distance, Art. 51, eq. (28),

$$c = \left(1 + \frac{h'}{h + 2h'}\right)\frac{h}{3} = \left(1 + \frac{2}{9}\right)\frac{20}{3} = 8.15 \text{ ft.}$$

$$\therefore\ Ec = 10{,}296 \times 8.15 = 83{,}900 \text{ ft.-lbs.}$$

In finding the center of gravity of any area, always divide it up into right triangles and rectangles. One side of each triangle is vertical and a vertical line through its center of gravity, cuts the horizontal base, ⅓ its length from the vertical side. To find how far the vertical through the center of gravity of the combined weight of wall above AB and of the earth over BC, acts from A, take moments about A.

Weight, Pounds	Arm, Feet	Moment, Foot-Pounds
25 × 150	$\frac{2}{3}(2.5) = \frac{5}{3}$	6,250
40 × 150	$2.5 + 1 = 3.5$	21,000
75 × 150	$4.5 + 2.5 = 7.0$	78,750
75 × 100	$4.5 + 5 = 9.5$	71,250
28,500		177,250

The line of action of the resultant weight is thus to the right of A, a distance,

$$177{,}250/28{,}500 = 6.22 \text{ ft.}$$

A good check is afforded by taking moments about some other point than A.

We next combine the weight above with E, acting 8.15 ft. above B. Suppose the resultant cuts the plane AB at a point distant x from A. Taking moments about this point; the moment of the resultant is zero ∴ the algebraic sum of the moments of its components is equal to zero.

$$\therefore\ 28{,}500\,(6.22 - x) = Ec = 83{,}900$$

$$\therefore\ x = 3.27 \text{ ft.}$$

Thus the base being 12 ft. in length, the resultant cuts it, 0.73 ft. outside of the middle third and $6 - x = 2.73$ ft. from the center of the base. The vertical component of this resultant = weight = 28,500 lbs., hence the unit pressures at A and B, are, Art. 15, eq. (3),

$$p = \frac{W}{l}\left(1 \pm \frac{6a}{l}\right) = \frac{28{,}500}{12}\left(1 \pm \frac{2.73}{2}\right).$$

$$\therefore \text{ stress at } A = 5{,}605 \text{ lb. sq. ft.} = 39 \text{ lb. sq. in.}$$

$$\text{stress at } B = -\,855 \text{ lb. sq. ft.} = -\,6 \text{ lb. sq. in.}$$

and about 20 in. of the base AB is under tension, varying from 0 to 6 lb. per sq. in.

The concrete is perfectly able to withstand this tension, but for a possible increase of the thrust from heavy rains and vibration, place $\frac{1}{2}''$ □ steel bars, say $3''$ from BC and $20''$ center to center. The bars, placed parallel to BC, may extend 2.5 ft. below AB to develop sufficient bond strength (50 diameters = 25″) and say, 4 feet above AB. The total tension on AB for 1 in. length of wall = $\frac{1}{2} \times 6 \times 20 = 60$ and on 20 in. length of wall (the spacing of the bars) = 1200 lb. The allowable stress on ½″ □ bar = $\frac{1}{4} \times 16{,}000 = 4000$ lbs., so that the rods are seen to be ample to take all the tension without entering into a precise computation. If AB is diminished in length the amount of tension increases and the wall approaches the T form, which is considered fully further on.

83. Design of Base Slab. We next consider the stability of the entire wall when the surcharge extends to N. Now, $h = 24$, $h' = 8$, ∴ as above,

$$E' = 0.143 \times 100\,(32^2 - 8^2) = 13{,}728 \text{ lbs.}$$

and the horizontal thrust E' acts above H,

$$c' = \left(1 + \frac{8}{24 + 16}\right)\frac{24}{3} = 9.60 \text{ ft.}$$

$$\therefore\ E'c' = 131{,}800 \text{ ft.-lbs.}$$

E' is the earth thrust on the plane HN and $E'c'$ is its moment about any point in LH.

The weight of the whole wall, with that of the earth over it, is next to be found, as well as its line of action.

The weight of footing $ALH = (4 \times 17 - \frac{1}{2} \times 4 \times 3)\,150$ . . = 9,300 lbs.
Weight of earth over $BM = 20 \times 100$ . . . . = 2,000 "
Weight of $ABCD$ and earth over it . . . . = 28,500 "
Total weight of wall and earth over it . . . . = 39,800 "

To utilize moments previously found, moments must still be taken about A.

Divide the footing cross-section into the rectangle AH and the trapezoid AL, which last, further subdivide into the rectangle $4' \times 1'$ and the remaining triangle.

Taking moments about A,

$$\left[13 \times 4 \times \frac{13}{2} - \left(1 \times 4 \times 2 + \tfrac{1}{2} \times 3 \times 4 \times \frac{4}{3}\right)\right]150. = 48{,}300 \text{ ft.-lbs.}$$

Mt. earth wt. over $BM = 20 \times 100 \times 12.5$ . . . . = 25,000 "
Moment previously found . . . . = 177,250 "
250,550 "

Thus the total weight of the entire wall and that of the earth directly over it, is $w' = 39,800$ lbs. and its moment about A is, 250,550 ft.-lbs. Its line of action is thus to the right of A, a distance,

$$\frac{250,550}{39,800} = 6.29 \text{ ft.},$$

and to the right of L, a distance, $4 + 6.29 = 10.29$ ft.

Let the resultant of E' and W' cut the base LH at a point x' ft. from L. Taking moments about this point,

$$W'(10.29 - x') = E'c'$$

or,

$$39,800\,(10.29 - x') = 131,800,$$

$$\therefore\ x' = 6.98 \text{ ft.}$$

The resultant thus cuts the base, $8.5 - 6.98 = 1.52$ ft. from the center of LH.

The unit stress on the earth foundation at L and H is,

$$p = \frac{39,800}{17}\left(1 \pm \frac{6 \times 1.52}{17}\right) = 2340\,(1 \pm 0.537)$$

$\therefore$ soil reaction at L = 3600 lb. sq. ft. = 25 lb. sq. in.
soil reaction at H = 1080 lb. sq. ft. = 7.5 lb. sq. in.

which is safe for ordinary (not soft) clay soils or for soils of ordinary clay mixed with dry sand. The soil pressures at L and H may be increased by supposing the surcharge to extend up to C; but, as in this instance, the increase will most probably not be enough to make the maximum soil pressure over 5000 lb. per sq. ft., which is allowed, the possible increase was not computed.

It is well to lay off the upward soil pressures at L and H, 3600 and 1080 lb. sq. ft. respectively, connect with a straight line and measure to scale the soil pressure vertically under A. It is found to be 3000 lb. sq. ft. The total soil pressure on the base of the toe AL is thus $\frac{1}{2}(3600 + 3000) \times 4 = 13,200$ lbs. (for one foot length of wall) and it acts through the center of gravity of the trapezoid or 2.1 ft. to the left of A.

In finding this distance, we can proceed either graphically (see Art. 34) or analytically. In the latter case, divide the trapezoid, having the vertical sides 3600 and 3000, into a rectangle of area $4 \times 3000 = 12{,}000$ and a right triangle of area $\frac{1}{2}(600) \times 4 = 1200$, the sum being 13,200.

Let y be the horizontal distance from the vertical through the center of gravity of this trapezoid to A. Taking moments about A,

$$13{,}200\, y = 12{,}000 \times 2 + 1200 \times \tfrac{2}{3} \times 4 = 27{,}200$$

$$\therefore y = 2.07, \text{ or say, } 2.1 \text{ ft.}$$

The work has been given in detail, in this example, to give the method to follow in all subsequent examples, where only results will be stated.

Let us investigate the strength of the toe AL, at the vertical section through A. Suppose reinforcing rods to be placed, as shown by the dotted lines, 3″ above the base; then the depth $d = IN$ (of the formulas of the Appendix) = 45″. The bending moment at A due to soil pressure is,

$$M = 13{,}200 \times 2.1 = 27{,}700 \text{ ft.-lbs.} = 332{,}400 \text{ in.-lbs.}$$

The percentage of steel to assume for such a deep section is very small—under 0.2% perhaps; hence from the diagram, App., Fig. 9, for $\beta = 36° \, 52'$, the inclination of the upper surface of the toe to the horizontal, we find $j > 0.92$. Take $j = 0.92$ $\therefore jd = 0.92 \times 45 = 41.4$ and assuming $f_s = 16{,}000$ lb./in.² in formula (14) of the Appendix, $f_s A j d = M$

$$\therefore 16{,}000 \times 41.4\, A = 332{,}400;$$

whence $A = 0.5$ sq. in. for $b = 12''$.

As in the case of deep sections, the bond stress is often large with large bars, try ½″ □ bars, 6″ center to center (abbreviated hereafter to 6″ c to c or 6″ ¢.), corresponding to $A = 2\left(\frac{1}{2}\right)^2 = 0.5$ sq. in., for $b = 12''$. The perimeter of one bar $= 2 = o_1$ $\therefore$ for $b = 12''$, the length of wall taken, $\Sigma o_1 = 4$.

By eq (34), App., the bond stress is given by,

$$(j\, d\, \Sigma o_1)\, u_1 = Q - \frac{M}{d} \tan \beta$$

$$= 13{,}200 - \frac{332{,}400}{45} \times 0.75 = 7660,$$

whence, $$u_1 = \frac{7660}{0.92 \times 45 \times 4} = 46.3 \text{ lb./in.}^2$$

Also, since $u_1 = \frac{v b}{\Sigma o}$ $\therefore v = \frac{u_1 \Sigma o_1}{b}$

whence the horizontal or vertical shear between the steel and the neutral axis, is,

$$v = \frac{40.3 \times 4}{12} = 15.4 \text{ lb. in.}^2$$

The values of u_1 and v are well below the limits 80 and 40 lb/in^2, so that larger bars can be used if desired. The $\frac{1}{2}''$ □ bars should extend from near L, over 50 diameters to the right of the vertical through A, or say $2' 6''$ to right of A to give sufficient embedment.

The value of f_c as computed from App. eq. 15, is 195 lb./in.^{2} For such deep sections, there is no need to compute f_c, as it will always be less than the allowable stress, 650 lb./in.^{2}

84. Resistance to Sliding. This subject, which should generally be treated earlier, has purposely been deferred, in order to make a number of comments, which will apply in other designs.

Calling f', the coefficient of friction of masonry on earth, the product of the total weight resting on the foundation by f' must equal the earth thrust on HN for exact equilibrium.

$$\therefore 39{,}800 f' = 13{,}730 \quad \therefore f' = 0.345.$$

Now f' for moist clay is given as 0.33; hence sliding would occur on such a foundation or on wet clay or quicksand. This may be obviated by increasing the amount of masonry, by inclining the foundation, or often, by adding a projection to the footing, below it, and near H. In the last case, the wall cannot slide without the projection pushing the earth before it; thus substituting friction of earth on earth for that of masonry on earth which is generally less. Of course the earth in front of the wall assists materially, when it is well compacted after the filling of the trench, but it is safer not to allow for this in the computation.

For dry earth or clay, f' can be taken as 0.5. Call σ the factor of safety against sliding.

$$\therefore \sigma \times 13{,}730 = 0.5 \times 39{,}800 = 19{,}900$$

$$\therefore \sigma = 1.45$$

This is perhaps satisfactory. In fact, if one could be assured that during heavy rains, f' should never fall below 0.5, the depth of footing could be decreased. However, it is wise not to pare down a wall to a very minimum, since w, φ and f', *vary* according as the earth is dry or saturated and our assumptions too, as to their values and consequently the value of the thrust may be far from the truth.

85. Approximate Method of finding Length of Base of T or Counterforted Walls. Let, Fig. 47, $ABCD$ represent the base slab and FI the vertical stem of either a T or a counterforted wall. IN is the level earth surface on which there may be a surcharge of height NN', which may be supposed to reach to either N or I.

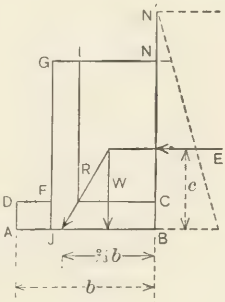

FIG. 47

Let E = earth thrust on vertical plane BN, which acts a
distance c above B.
b = length of base AB.

In computing the weight resting on the base AB, ignore the weight of toe AF and for the case where the surcharge only extends to N, replace the weight of the remainder of the wall and the earth over it by that of a rectangular prism of earth of height BN, width GN and length 1 foot perpendicular to the plane of paper.*

Let W = weight of this rectangular prism of earth BG. Its line of action is thus, $\frac{1}{2}\,GN$ to left of C. Then if R, the resultant of E and W, is assumed to cut AB, $\frac{2}{3}\,b$ from B, the moment of W about this point, will equal the moment Ec of E about the same point.

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* This device was suggested by an article on the "Design of Standard Reinforced-concrete Retaining Walls" in *Engineering News*, July 24, 1913, by Mr. H. M. Gibb, who uses it in deriving a number of interesting conclusions. The resultant on the base (of length b) always passing through the outer third point, it was found, as the vertical slab was moved back from the toe towards the heel, (1) that the unit toe pressures decrease, (2) the resistance to both overturning and sliding decreases, (3) that b is a minimum when the length of toe = $\frac{1}{3}\,b$, but that there is only 3% difference as the length of toe varies from $\frac{1}{6}\,b$ to $\frac{1}{2}\,b$.

Before stating this equality, the desired length of toe DF, must be expressed in terms of b. The moment equation will then determine b.

Ex. 1. Make an approximate computation of the length of base b of the T wall, Fig. 48, Art. 86, the surcharge not extending to left of N. From the figure and the computation for E and c in Art. 86, we have for an assumed length of toe $DF = 0.36\,b$, $GN = 0.64\,b$, Fig. 47; $\therefore$ W = weight of earth BG (the earth weighing 100 lb./ft.^{3}) $= 0.64\,b \times BN \times 100 = 64\,b \times 17 = 1088\,b$·lbs., and it acts $\frac{1}{2}(0.64\,b) = 0.32\,b$ to left of B and $0.667\,b - 0.32\,b = 0.347\,b$ to right of point where R cuts AB, whence equating moments of W and E about this point,

$$1088\,b \times 0.347\,b = 8020 \times 7.04 = 56{,}460$$
$$\therefore b^2 = 150 \;\therefore\; b = 12.25', \; DF = 0.36\,b = 4.4'$$

It will be observed that in Fig. 48, b was assumed 12.5 ft., length of toe, 4.5 ft. and in Fig. 48(a) (case a or surcharge extending only to N) the resultant cuts the base slightly within the outer third limit. The approximation is thus remarkably close and it is equally so in the next example.

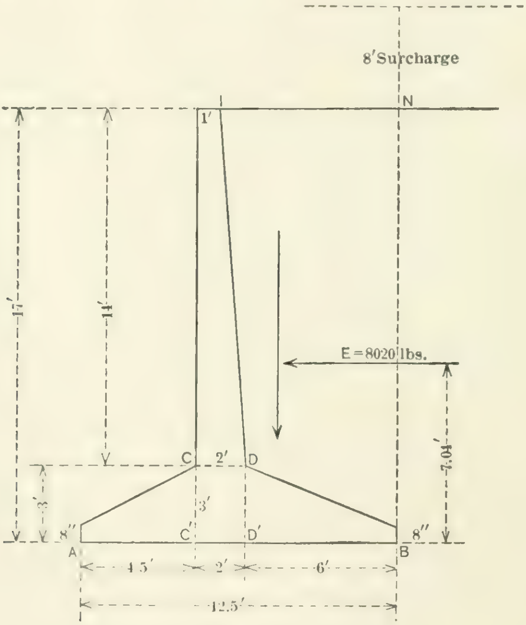

Fig. 48

Ex. 2. Refer to Fig. 50, Art. 86(a) for quantities. Assume length of toe $= 0.3\,b$ $\therefore$ $GN = 0.7\,b$. Hence for surcharge extending to wall, the weight W will be assumed to be the weight of a prism of earth BG, Fig. 47, plus the weight of surcharge, 8′ high, over GN or the weight $BN' \times GN \times 100$

$= 25 \times 0.7\,b \times 100 = 1750\,b$ pounds. Here, W acts to left of B, $\frac{1}{2}\,(0.7\,b) = 0.35\,b$ and to right of the point $\frac{2}{3}\,b$ from B or where R is assumed to cut the base $(0.667 - 0.35)\,b = 0.317\,b$. Whence, taking moments about this point

$$1750 \times 0.317\,b^2 = E\,c = 8020 \times 7.04 = 56{,}460$$
$$\therefore\ b = 10.1', \quad DF = 0.3\,b = 3.03 \text{ ft.}$$

which agree almost exactly, with the values adopted, which correspond to a resultant on the base cutting it at the outer third point. With the surcharge extending only to N, it will be found that $b = 12.25'$ in place of $10.1'$.

Ex. 3. Consider the counterforted wall, Fig. 52, Art. 88, where E for 1 ft. length of wall = 14,660 lbs. and $c = 9.96$ ft. With the lettering of Fig. 47, assume $DF = 0.3\,b$ $\therefore$ $GN = 0.7\,b$ and for the surcharge extending only to N, $W = 0.7\,b \times BN \times 100 = 70 \times 25\,b = 1750\,b$ lbs. and its line of action is $0.35\,b$ from B or $(0.667 - 0.350)\,b = 0.317\,b$ from the point where R cuts the base, $\frac{2}{3}\,b$ from B.

Equating moments about this point,

$$1750\,b \times 0.317\,b = Ec = 146{,}000$$
$$\therefore\ b^2 = 146{,}000 \div 555 = 263\ \therefore\ b = 16.2 \text{ ft.,}$$

and $DF = 0.3\,b = 4.86$ ft.

The dimensions actually adopted, satisfying the outer third condition, are $b = 15.5$ ft., $DF = 4.5$ ft. = length of toe.

It is seen from these examples that the approximate method gives rather close results. The reason is (see Fig. 47) that although W is always less than the true weight, its arm about the outer third point is greater than the true arm, so that the product is not greatly altered.

86. T Wall with Surcharge.

The length of base shown in Fig. 48, was based upon the preliminary investigation of Art. 85, Ex. 1.

MASONRY

($w' = 150$ lb. cu. ft.)

Take moments about C

Section	Area, Square Feet	Arm, Feet	Area Moment
Above CD	$14 \times 1 = 14.$	0.5	7.
	$7 \times 1 = 7.$	1.33	9.33
Below CD	$2.33 \times 2 = 4.67$	1.	4.67
	$2.33 \times 3 = 7.$	4.	28.
	$0.67 \times 12.5 = 8.33$	1.75	14.58
	$2.33 \times 2.25 = 5.25$	−1.5	−7.87
	46.25		55.71

Thus the weight of masonry, per foot of length, is $46.25 \times 150 = 6937$ lbs. and its moment about C is, $55.71 \times 150 = 8355$ ft.-lb.

EARTH FILLING

(w = 100 lb. cu. ft.)

Take moments about C

Section	Area, Square Feet	Arm, Feet	Area Moment
Above D	$7 \times 1 = 7.$	1.67	11.67
	$14 \times 6 = 84.$	5.	420.
Below D	$2.33 \times 3 = 7.$	6.	42.
	98.		473.67

Hence the weight of earth over the wall, per foot of its length, is 9800 lbs. and its moment about C is, 47,367 ft.-lbs.

Case (a). With no surcharge over IN:

	Weight, Pounds	Moment, Ft.-Lbs.
Masonry	6,937	8,355
Earth	9,800	47,367
	16,737	55,722

Hence the combined weight is 16,737 lbs. and it acts 55,722/16,737 = 3.34 ft. to right of C.

Case (b). Surcharge over IN. The weight of the surcharge is $7 \times 8 \times 100 = 5600$ lbs. and its moment about C is $5600 \times 4.5 = 25{,}200$ ft.-lbs. Add these to the preceding totals. The total weight of masonry, earth and surcharge is 22,337 lbs. and its total moment about C is 80,922 ft.-lbs. The resultant weight thus acts 80,922/22,337 = 3.63 ft. to right of C. Railway tracks, 13′ c to c, will be assumed, parallel to the wall. As such tracks may be very near the wall and one or more may be occupied by locomotives, both cases (a) and (b) of loading and the resulting stresses should be investigated.

Resistance to sliding. Let $f' = 0.5$.

$$\text{Case } (a), \text{ factor safety} = \frac{0.5 \times 16{,}737}{8020} = 1.04$$

$$\text{Case } (b), \text{ factor safety} = \frac{0.5 \times 22{,}337}{8020} = 1.39$$

Face Slab CI. The height of face slab is 14 ft. Let h = depth from I (top of wall) to point considered. Taking $h = 15$ gives a thrust in excess,

$$E = 14.3\,(23^2 - 8^2) = 6650 \text{ lbs.}$$

and its resultant acts above the level $h = 15$,

$$c = \left(1 + \frac{8}{31}\right)\frac{15}{3} = 6.30 \text{ ft.}$$

At $h = 10$, $E = 14.3\,(18^2 - 8^2) = 3720$ lbs.

It acts above the level $h = 10$,

$$c = \left(1 + \frac{8}{26}\right)\frac{10}{3} = 4.36 \text{ ft.}$$

Similarly at $h = 5$, $E = 14.3\,(13^2 - 8^2) = 1500$ lbs. and it acts,

$$c = \left(1 + \frac{8}{21}\right)\frac{5}{3} = 2.30 \text{ ft. above } h = 5.$$

Hence the bending moments in the face slab,

at $h = 15$, $= 6650 \times 6.3 \times 12 = 502{,}740$ in. lbs.
at $h = 10$, $= 3720 \times 4.36 \times 12 = 194{,}600$ in. lbs.
at $h = 5$, $= 1500 \times 2.3 \times 12 = 41{,}400$ in. lbs.

These moments, like the thrusts, correspond to 1 ft. length of wall $\therefore b = 12''$.

Consider horizontal sections of the face slab at these depths, for which $b = 12''$ and d = thickness of slab from the front face to the center of the steel bars placed parallel to and 2.5″ from ID. The values of d at $h = 5, 10, 15$ ft. are 13.5, 17.5, 21.5 inches, respectively. Since the rear face makes only a small angle with the exposed face, the formulas and diagrams for prismatic beams will be used, Appendix, eqs. (18), (19), (20), (21) and Fig. 11.

Thus at $h = 15$, $R = \dfrac{M}{bd^2} = \dfrac{502{,}740}{12 \times (21.5)^2} = 90.5$

$h = 10$ $R = \dfrac{M}{bd^2} = \dfrac{194{,}600.}{12 \times (17.5)^2} = 53$

$h = 5$, $R = \dfrac{M}{bd^2} = \dfrac{41{,}400}{12 \times (13.5)^2} = 19.$

From the diagram, Fig. 11, with $R = 90$, $f_s = 16{,}000$, we find $f_c = 575$, $p = .0063$, $j = 0.88$ $\therefore$ steel area at $D = p\,b\,d = 0.0063 \times 12 \times 21.5 = 1.62$ sq. in. Take ¾″ □ bars, 4″ c to c, giving $A = 1.68$ sq. in. for $b = 12''$.

To test the shearing and bond resistance at D, notice that in formula (35) of the Appendix, the shear Q = earth thrust.

$$\therefore \text{ at } h = 15,\ v = \frac{Q}{bjd} = \frac{6650}{12 \times 0.88 \times 21.5} = 29 \text{ lb. sq. in.;}$$

$$u = \frac{Q}{jd\,\Sigma o} = \frac{6650}{0.88 \times 21.5 \times 3 \times 3} = 39 \text{ lb. sq. in.}$$

Both are well within limits prescribed. We proceed similarly at $h = 10$.

With $R = 53$ and $f_s = 16{,}000$, we find from the diagram, $f_c = 420$, $p = 0.0035$.

Since p is about half the previous value, cut out one-half the bars, $\therefore$ use ¾″ □ bars, 8″ c to c. This gives $p = \frac{9}{16} \div (8 \times 17.5) = .0040$. With this value of p and $R = 53$, the diagram gives $f_s = 14{,}600$, $f_c = 400$, $j = 0.90$.

For shear, $$v = \frac{Q}{bjd} = \frac{3720}{12 \times .90 \times 17.5} = 19.7 \text{ lb. sq. in.}$$

For bond stress, $$u = \frac{Q}{jd\Sigma o} = \frac{3720}{.90 \times 17.5 \times 4.5} = 52.5 \text{ lb. sq. in.}$$

Both are well within the limits allowed. Lastly at $h = 5$, place ¾″ □ bars, 16″ c to c. $\therefore$ $p = \frac{9}{16} \div (16 \times 13.5) = .0026$, and with $R = 19$, we find from the diagram,

$$f_s = 8000,\ f_c = 200,\ j = 0.93$$

$$\therefore v = \frac{1500}{12 \times .93 \times 13.5} = 10 \text{ lb. sq. in.}$$

$$u = \frac{1500}{.93 \times 13.5 \times 2.25} = 53 \text{ lb. sq. in.}$$

Both safe values.

These (nearly) vertical bars will be hooked in the projection at the bottom of the footing.

Soil Pressures. Recalling that in case (a), the resultant of the weight of wall and earth over it, $W_a = 16{,}737$ lbs., acts 3.34 ft. to right of C, and in case (b), the resultant, $W_b = 22{,}337$ lbs. acts 3.63 ft. to right of C, we proceed next to combine W_a and W_b in turn, with the earth thrust $E = 8020$ lbs. on BN, which acts 7.04 ft. above B

Case (a). Let the resultant R_a of E and W_a, cut AB at F_a (not shown on the figure) and call the arm of W_a about $F_a = d_a$. Take moments about F_a.

$$W_a d_a = E \times 7.04$$

$$\therefore d_a = \frac{56{,}460}{16{,}737} = 3.38 \text{ ft.}$$

Thus R_a cuts the base at F_a, $3.38 - 3.34 = 0.04$ ft. to left of C, or $a = 1.79$ ft. from the center of the base $\therefore$ in the formula for soil pressures, $l = 12.5'$, $a = 1.79'$,

$$\therefore p = \frac{16{,}737}{12.5}\left(1 \pm \frac{10.74}{12.5}\right) = 1340\ (1 \pm 0.86)$$

$\therefore$ soil pressure at A = 2490 lb. sq. ft.
soil pressure at B = 190 lb. sq. ft.

Case (b). Let the resultant R_b of E and W_b, cut the base at F_b, the arm of W_b about F_b being d_b. Take moments about F_b and solve for d_b.

$$\therefore d_b = \frac{56{,}460}{22{,}337} = 2.53 \text{ ft.}$$

Thus R_b cuts AB, $3.63 - 2.53 = 1.10$ ft. to right of C, or $4.5 + 1.1 = 5.6$ ft. to right of A or $6.25 - 5.60 = 0.65$ ft. to the left of the mid-point of AB.

$\therefore$ in the formula, $a = 0.65$, $l = 12.5$

$$p = \frac{22{,}337}{12.5}\left(1 \pm \frac{3.90}{12.5}\right) = 1790\ (1 \pm 0.312)$$

$\therefore$ soil pressure at A = 2345 lb. sq. ft.
soil pressure at B = 1230 lb. sq. ft.

These soil pressures (or reactions) are plotted in Fig. 48*a* and the resultant reactions on heel and toe, in position and magnitude, are found as hitherto explained and marked on the figure. The weight of the heel (6 ft. base) and earth over it,

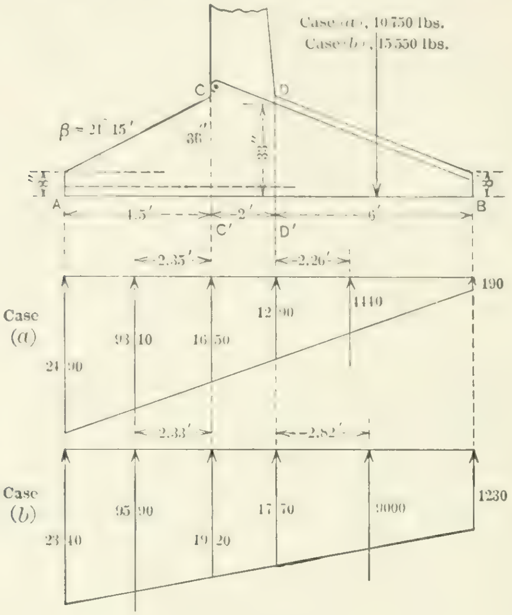

FIG. 48(*a*)

case (*a*) is 10,750 lbs., for case (*b*), 15,550 lbs. The resultant of each weight will be regarded as acting along a vertical through the mid-point of $D'B$.

Heel. The heel $DD'B$ will be treated as a cantilever beam, subjected to the action of its weight, with that of the earth over it and the soil reaction on $D'B$, the breadth of beam, perpendicular to plane of paper, being $b = 12$ inches.

Take moments about D'

Case (*a*), $M_a = 10{,}750 \times 3 - 4440 \times 2.26 = 22{,}216$ ft. lbs.

Case (*b*), $M_b = 15{,}550 \times 3 - 9000 \times 2.82 = 21{,}270$ ft. lbs.

M_a is the greater and will be used in designing the steel-reinforcement, which is to be placed 3″ from the upper face of the heel.

At the section DD', d = distance from D' to steel, measures $33''$, also $\beta_1 = 21^\circ\,15'$, $\cos\,\beta_1 = 0.932$, β_1 being the angle the rod makes with the normal to the section DD' or with the horizontal.

By App., eq. (11), $M = f_s\,A_1 . jd . \cos\,\beta_1$; whence assuming $j = 0.9$, $f_s = 16{,}000$,

$$22{,}216 \times 12 = 16{,}000\,A \times 0.9 \times 33 \times 0.932$$

$$\therefore\; A = 0.604 \text{ sq. in.}$$

If we take 5/8″ □ deformed bars, 6″ c to c, $A = 0.784$,

$$\text{whence, } p = \frac{0.784}{33 \times 12} = 0.002.$$

From Fig. 9, App., dotted lines, we find for $p = 0.002$ and $\beta_1 = 21^\circ\,15'$, $k = 0.211$, $j = 0.93$, so that using $j = 0.9$ above, was on the side of safety.

The bars are to be extended about 40 diameters $= 40 \times 5/8 = 25''$ into the face slab and hooked over a horizontal bar at the end. This provides sufficient bond resistance in the vertical slab.

As it is not always possible to know beforehand the character of the foundation, it is the practice of some to omit the soil reaction in the above computation. With a yielding foundation, if the wall has moved over at the top so much that there is no soil reaction over $D'B$, not only the weight of heel and earth over it, acts on the heel, but in addition there is the friction due to the earth thrust on BX. The latter is nearly equal to Ef ($f = 2/3$ in this case) and acts downward along XB. If the foundation is doubtful, all of these forces should be included in designing the reinforcement. In the Appendix, Art. 10, is given an illustration of this method of computation.

There is sometimes a difficulty in securing sufficient bond resistance, due to the short extension available of the inclined bar into the face slab. This may be met by bending the bar downward across the stem and into the toe.

Maximum unit shear and bond stress at DD'. The external shear is greatest for case (*b*) and is,

$$Q = 15{,}550 - 9000 = 6550 \text{ lbs.}$$

since $\beta = 0$, formulas (20) and (33), App., reduce to,

$$u_1 . jd\Sigma o_1 = Q - \frac{M}{d} \tan \beta_1 = vb . jd.$$

Ignoring the term $M \tan \beta_1/d$, the resulting values of u_1 and v will be in excess.

$$\text{Shear } v = \frac{Q}{b . jd} = \frac{6550}{12 \times 0.93 \times 33} = 18 \text{ lb./in.}^2$$

With 5/8″ ☐ bars, 6″ c to c, $o_1 = 2.5$ in 6 in. ∴ $\Sigma o_1 = 5$ in 12 inches. Hence the bond stress

$$u = \frac{Q}{jd\, \Sigma o_1} = \frac{6550}{0.93 \times 33 \times 5} = 42.6 \text{ lb./in.}^2$$

Hence, both shear and bond stresses are safe. The unit stress f_c in the concrete at D', was computed and found to be small.

Toe AC′C. We have $\cos \beta = \cos 21° 15' = 0.932$, $\cos^2 \beta = 0.869$. Assume $k = 0.23$ ∴ $j = 0.92$, corresponding to $p = 0.002$ by Fig. 9, App., ∴ $kj = 0.212$.

As in the previous design, neglect the weight of toe for additional security and compute shear and moment at CC' from the soil reaction only. Both are greatest for case (*b*), for which $Q = 9590$ lbs. and $M = 9590 \times 2.33 \times 12 = 268{,}140$ in. lbs.

Assume horizontal bars, 3″ from AB and extending 7 ft. from A ∴ at CC', $d = 33''$. Assuming $f_s = 16{,}000$ lb./in.², by App., eq. (14), steel area

$$A = \frac{268{,}140}{16{,}000 \times 0.92 \times 33} = 0.554 \text{ sq. in.}$$

Here, either 3/4″ ☐ bars, 12″ c to c, or 5/8″ ☐ bars, 6″ c to c, will suffice. Let us test the latter for shear and bond stress. For the 5/8″ ☐ bars, 6″ c to c, $\Sigma o_1 = 2 \times 2.5 = 5$ sq. in. By App., eq. (34),

$$u_1 . jd\, \Sigma o_1 = Q - \frac{M}{d} \tan \beta = vb . jd.$$

Neglecting the term in M to give an excess.

$$\therefore \text{ bond stress } u_1 = \frac{9590}{0.92 \times 33 \times 5} = 63 \text{ lb./in.}^2$$

$$\text{unit shear } v = \frac{9590}{12 \times 0.92 \times 33} = 26.5 \text{ lb./in.}^2$$

As these are safe values, the true shear and bond stresses, which are less, are safe values.

For the $\frac{5}{8}''$ □ bars, 6″ c to c, $A = 0.784$ sq. in. $\therefore p = 0.784 \div (12 \times 33) = 0.002$, as assumed above. Also from App. eq. (15), it is found that $f_c = 224$ lb./in.^{2}

Let us now test the $\frac{3}{4}''$ □ bars, 12″ c to c, for bond and shear stresses. Here the term, $M \tan \beta / d$ must be included.

$$\text{We have, } \frac{M}{d} \tan \beta = \frac{268{,}140}{33} \times 0.389 = 3160 \text{ lbs.}$$

$\therefore u_1 . jd\ \Sigma o_1 = 9590 - 3160 = vb . jd.$

Since $\Sigma o_1 = 3$.

$$\therefore u_1 = \frac{6430}{0.92 \times 33 \times 3} = 70.7 \text{ lb./in}^2,$$

$$v = \frac{6430}{12 \times 0.92 \times 33} = 18 \text{ lb./in.}^2,$$

both safe values, so that $\frac{3}{4}''$ □ bars, 12″ c to c, can be used if desired.

Temperature Reinforcement. The area of *ICD*, Fig. 48, is $18 \times 168 = 3024$ sq. in. and $\frac{1}{3}$ of 1% of $3024 = 10.08$ sq. in. $22 - \frac{3}{4}''$ □ bars, give a total area = 12.4 sq. in. Hence for temperature reinforcement, use 14 horizontal $\frac{3}{4}''$ □ bars, 12″ c to c and $2\frac{1}{2}''$ from the exposed face and 8 horizontal $\frac{3}{4}''$ □ bars, 24″ c to c, resting against the vertical reinforcement of the vertical slab or 3″ about, from rear face.

The horizontal bars near the front face rest against $\frac{1}{2}''$ □ vertical bars, 30″ c to c. Also $\frac{1}{2}''$ □ horizontal bars, spaced 12″ c to c, will be placed longitudinally or parallel with the face wall under the inclined rods of the heel and over the horizontal rods of the toe.

In the final drawing, Fig. 49, all the rods are shown in position, and it will be noticed that a projection on the bottom of the footing has been added, not only to furnish additional bond resistance to the vertical reinforcement, but also to increase materially the resistance to sliding. Before sliding can occur, the projection will have to push the earth to the left. Hence for the 5.5 ft. from A to the projection, on which the soil reaction, case (a), is easily found from Fig. 49, to be 10,890 lbs., the coefficient of friction is that of earth on earth. Hence the resistance to sliding is $\frac{2}{3} \times 10{,}890 = 7260$ lbs. The soil reaction on the remainder of the base is 5847 lbs., and the total frictional resistance, for $f' = 0.5$, is $0.5 \times 5847 = 2923$ lbs. Thus the total resistance to sliding is $7260 + 2923 = 10{,}183$.

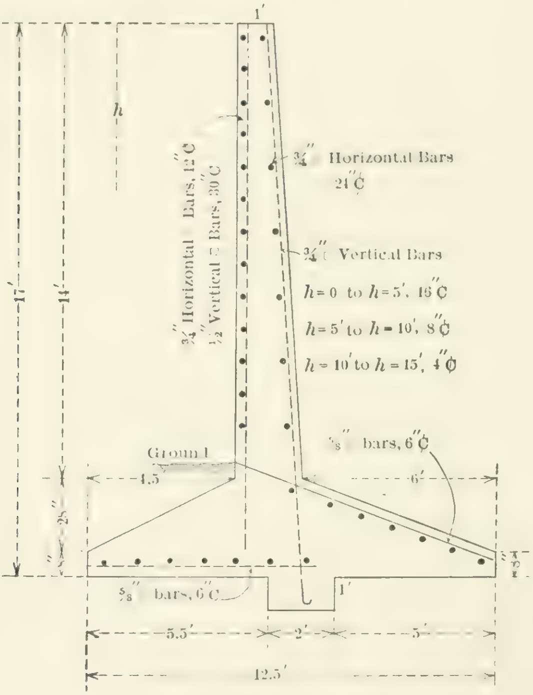

FIG. 49

Since the horizontal earth thrust = 8020 lbs., the factor of safety against sliding $= \dfrac{10{,}183}{8{,}020} = 1.27.$*

Thus the projection has increased the resistance to sliding by 22%. To be most effective, it should be placed as far to the rear as possible and not under the toe. The earth in front of the wall offers some resistance, especially if it is well rammed, but it is well to ignore it in the computation.

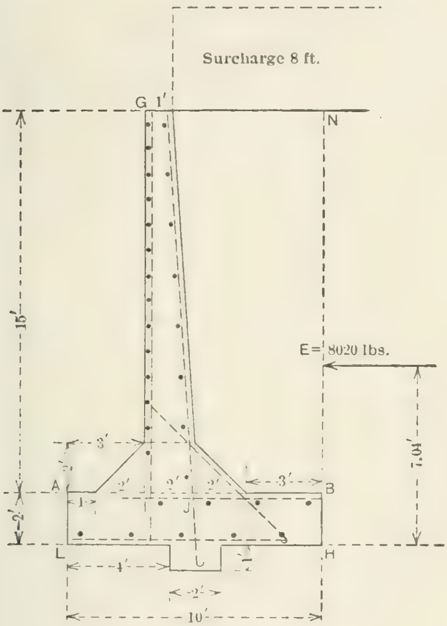

FIG. 50

It will be observed that the projection is under "punching" shear, where 120 lb. sq. in. is allowed. The resistance at a horizontal section is thus 24 × 12 × 120 = 29,760 lbs., which is much greater than the shearing force acting on it.

86 (a). *A second familiar type of T wall* is shown in Fig. 50, the total height of wall and surcharge being the same as for the type Fig. 48. The surcharge for this wall is supposed to correspond to a fixed load, as buildings, machinery, etc., and its weight is supposed to correspond to that of earth, 8 ft. high, extending up to the wall and fixed in position. The front face of the stem of the T is vertical, the thickness at top is 1 ft. and the rear face produced meets AB at J, 5 ft. to the right of A and 5 ft. to left of B. The preliminary computation for the length of base has been given in Art. 85, Ex. 2. The student can show that with the dimensions given in Fig. 50, the resultant on the base cuts it almost exactly $\frac{1}{3}$ LH from L, giving a vertical component of soil pressure at L, 3970 lb./ft.^{2} and at H, zero. The factor of safety against sliding, for $f' = 0.5$, is 1.24, not including the aid given by the projection.

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* This is doubtless too high; porous earth yields more readily than a solid to compression, and any sliding would be progressive, starting first at the projection.

The reinforcement throughout was placed 2.5 inches from the nearest face. That for the vertical slab is precisely the same as for the preceding design, the stem being practically the same. The reinforcement for the fillets and base slab are treated in full in the Appendix, Arts. 11 and 18.

If preferred, the upper reinforcement of the rear fillet can be bent downward across the stem and be made to pass, say $2\frac{1}{2}''$, from the upper surface of the front fillet, thus materially ensuring it against shrinkage cracks. Similarly, it is sometimes the practice to bend some of the vertical bars of the rear face of the vertical slab across the base slab, the continuation extending, say 2.5 ins., from LH and constituting the lower reinforcement of the base slab.

87. T Wall. Surface of Earth Sloping Upward. When the surface of the earth slopes indefinitely upward at any angle less than or equal to φ, the earth thrust on BN, Fig. 51, will act parallel to the top slope. Next the face slab, since the earth generally settles relatively to the wall, friction of earth on wall will be exerted, and the earth thrust will be changed in amount and direction, except for the case where the surface slopes at the angle of repose. For simplicity, suppose, for any inclination of the surface, that the earth thrust, for the same depth, remains the same in amount and direction as the wall is approached.

In Fig. 51, draw $D'M$ parallel to IN and let E' = earth thrust on MN = thrust on $D'I$, E'' = earth thrust on BM, both acting parallel to the surface IN.

$\therefore$ E = earth thrust on $BN = E' + E''$.

Also, let W' = weight of wall.

W'' = weight of earth over heel.

When E' acting on ID' and E'' acting on BM are combined with W', the resultant on the base is the same as the resultant of $E = E' + E''$, acting on BN and W'. Such a resultant, for usual designs, will cut AB so near A that the wall will tend to move over at the top. This tendency is counteracted by W'', which thus inevitably comes into play, so that the true resultant on the base is found by combining E with W' and W''. Thus the usual procedure is justified. A special case is when IN is horizontal, as in previous examples.

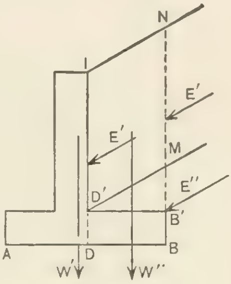

Fig. 51

For the T wall of Fig. 51, the *face slab* is designed as before, using the horizontal component of E' and ignoring the vertical component. The *soil reactions* are found as usual, as soon as the resultant of E, W' and W'' is found and the point where it cuts the base. Then the *toe* is designed as hitherto explained.

For the *heel*, note that the inclined thrust E' on MN is transmitted unchanged to $D'I$. Its vertical component is *not* then transmitted to B if the wall and foundation are absolutely unyielding. The thrust E'' is carried directly to the heel and its moment about the mid-point of the section DD', must be added algebraically to the moments due to the total soil reaction on DB, the weight of the heel $D'B$ and the weight W'' of the earth over it. But since the wall and its foundation are not rigid, there will be a moving over of the wall from the more or less elastic yielding of wall and foundation, so that a certain unknown amount of friction will be exerted along BN, tending to counteract the motion. The full amount of friction that can be exerted is approximately f times the horizontal component of E, and all of this will come into play if the wall is on the point of overturning.

As the foundation is often doubtful, it seems best to design the heel for this extreme case, and it is suggested that the reinforcement at DD' be figured from the three vertical forces: the vertical component of the soil reaction on DB, the weight W'' and the full friction that can be exerted along NB. This neglects the friction on the base and the horizontal component of E'', which is on the side of safety.

It may be thought that the horizontal component of E, acting to the left, at a distance $\frac{1}{3}$ BN above B, should be included in the computation, but it must be borne in mind that the earth over the heel is not a solid body, rigidly connected with the heel, but a granular mass; so that the only way to transfer even the horizontal component of the earth thrust acting on $B'M$ is by friction, exerted along the top of the heel, between the heel and the earth over it. Also, by assumption the horizontal component of E', acting on MN, is supposed transmitted to $D'I$; hence it should be omitted from the forces assumed to act on the he

Since it cannot be known in advance how much friction is exerted on BN, it is sometimes omitted, as well as the soil reaction, so that the heel is designed simply for the force W''.

Railroad Embankment. When the T wall sustains a railroad embankment, sloping up from the top of the wall, the surface at a certain height being level to accommodate tracks, only an approximate solution can be indicated. The tracks will be supposed to carry trains, or for computation, an equivalent weight of earth. Then the construction of Art. 32 is used to find the thrust on BN and MN, assuming an inclination to the thrust intermediate between IN and the horizontal. For safety, suppose any thrust to act $\frac{4}{10}$ height above the base. See Art. 43

The earth thrust on $D'I$, Fig. 51, will be taken as the same in amount and direction as that on MN. The design of stem, toe, and heel then proceeds exactly as before. The inclination of the thrust on BN or MN can not be indicated exactly. For no surcharge the thrust is horizontal; for a high embankment

(100 to 150 ft.) and low retaining wall (10 to 20 ft.) the thrust makes nearly the angle φ with the horizontal. For intermediate heights, its inclination lies between 0 and φ. A safe value can be chosen. For very high embankments, the solution approaches that for an indefinitely great surcharge.

88. Counterforted Walls. For retaining walls of heights greater than about 17 ft., counterforted walls are found to be more economical than T walls.

In Fig. 52 is shown this type of wall in plan and elevation. In the elevation, FI is the face slab, AF the toe, DB the heel slab, with the "projection" extending 2 ft. below B. The counterforts DCI are 18 ins. wide and 10 ft. apart center to center. A coping should be added at the top of the exposed face, in this and previous walls, but it is purposely omitted to simplify computations. The dimensions, shown fully on the figure, were only arrived at after a preliminary design with a shorter base was investigated and found insufficient, giving too low a resistance to sliding and too large soil pressure at the toe. Also see Art. 85, Ex. 3. The thickness of face slab was arbitrarily taken at 12 ins., since thinner walls are not so easy to pour.

A surcharge of loads equivalent to 8′ depth of earth is assumed. The wall will be designed for reinforcement for the same working stresses and data used, as in previous computations: $f_s = 16{,}000$ lbs. sq. in., $f_c = 650$ lbs. sq. in., $v = 40$ lbs. sq. in., $u = 80$ lbs. sq. in. for plain bars, $n = 15$, $\varphi = 33^\circ 41'$, $w = 100$ lbs. cu. ft., $w' = 150$ lbs. cu. ft., soil pressure = 5000 lbs. sq. ft., and the coefficient of friction f' of masonry on earth = 0.5.

Face Slab. The face slab, of course, acts as a whole, but for purposes of computation it will be supposed divided into a series of independent continuous horizontal beams, each 12″ high, 10′ span (c to c of counterforts), with bending moments at center and at junction with counterforts of $\frac{1}{12} pl^2$ and $-\frac{1}{12} pl^2$ (in ft. lbs.), where p = horizontal thrust of the earth in lbs. per sq. ft., at the depth h from top of wall to center of beam and l = length of beam = 10 ft. This is decidedly on the side of safety near the junction of the face slab with the footing,

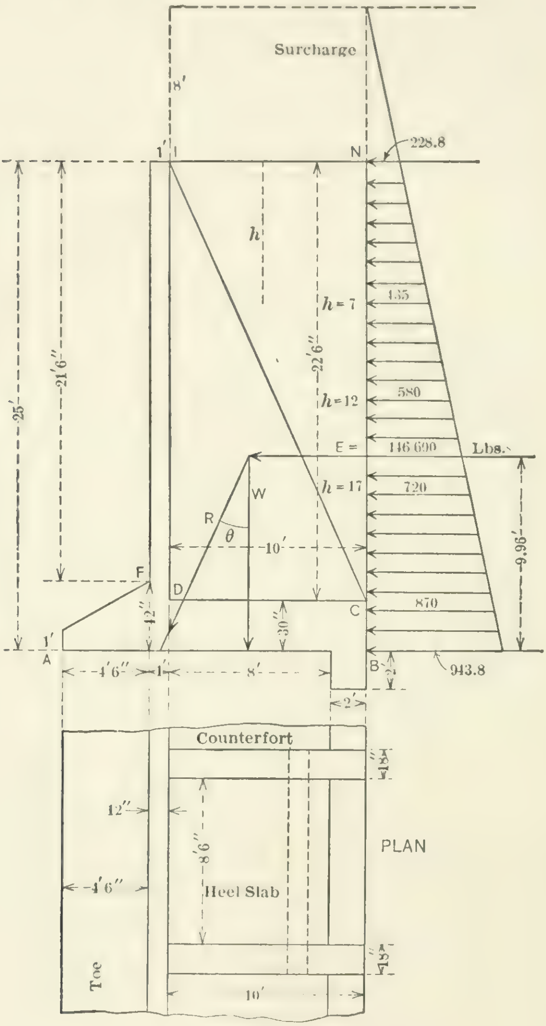

FIG. 52

which is not undesirable. The reinforcing bars will be placed 2″ from the front face for its whole length and other bars will be placed 2″ from the rear face at the counterforts and extending to ¼ span on either side, to take care of the negative moments.

The surcharge being 8 ft., the unit pressure p at any depth h below the top of the wall is, Art. 51,

$$p = w\,(8 + h)\,tan^2\left(45° - \frac{\varphi}{2}\right) = 28.6\,(8 + h).$$

The moment at center and ends of beam is, in amount,

$$M = \frac{1}{12}\,pl^2 \times 12 = 100\,p, \text{ in. lbs.}$$

It will be found that the spacing of the horizontal reinforcement is really determined by the bond stress, as given further on; hence the approximate formula for A was used.

$$M_s = f_s\,j\,d\,A = 16{,}000 \times \tfrac{7}{8}\,d\,A = 14{,}000\,d\,A.$$

$$\therefore\; A = \frac{M}{14{,}000\,d}$$

where $d = 12 - 2 = 10$ in. throughout.

The size and spacing of the bars, as determined from this formula, are given in the following table:

HORIZONTAL BARS IN VERTICAL SLAB

Depth h, Feet	p, Pounds Sq. Ft.	$M = 100\,p$ Inch-Pounds	Reinforcement			
			$A = \dfrac{M}{14000d}$	Size	Spacing	Adopted Spacing
2	286	28,600	0.204	½″ □	14″ ¢	8″ ¢
7	429	42,900	0.307	½″ □	10″ ¢	8″ ¢
12	572	57,200	0.409	½″ □	7″ ¢	6″ ¢
17	715	71,500	0.511	½″ □	5″ ¢	5″ ¢
22	858	85,800	0.613	½″ □	4″ ¢	4″ ¢

Spacing for Bond Stress. Taking the working value for bond stress, $u = 80$ lbs. sq. in., by App., Art. 20, the spacing in inches is, for $b = 12''$,

? see p. 271

$$x = \frac{960\,jd}{Q} \quad (0)$$

where Q = external shear.

For a ½″ □ bar, perimeter = (o) = 2; also $jd = \frac{7}{8}(10)$

$$\therefore x = \frac{16{,}800}{Q}$$

Q = shear = reaction of beams 10′ long, 1′ high

$= \frac{1}{2} p \times 10 = 5p$.*

h. Ft.	$Q = 5p$ Lbs.	$x = \frac{16800}{Q}$ In.
2	1,430	12″ ¢
7	2,145	8″ ¢
12	2,860	6″ ¢
17	3,575	5″ ¢
22	4,290	4″ ¢

Some of these values being less than those found above were adopted, as indicated in the table above.

The values of f_c for any h are all less than 650 lbs. sq. in. as can be shown most easily by aid of App., Fig. 11.

$$\text{Thus, } R = \frac{M}{bd^2} = \frac{M}{12 \times 10^2} = \frac{M}{1200}.$$

Hence at $h = 22$, $R = 71.5$ and with $f_s = 16{,}000$, the diagram gives, $p = 0.005$, $f_c = 500$ lbs. sq. in., $j = 0.89$ $\therefore A = p.bd = 0.600$ sq. in. The maximum shear is at $h = 22$ and is,

$$v = \frac{Q}{bjd} = \frac{4290}{12 \times 9} = 40 \text{ lb. sq. in.}$$

which is allowable. See p. 164, for increase in steel area due to temperature stresses.

Earth Thrust on 10 ft. Length of Wall. The horizontal earth thrust on BN, for 10 ft. length of wall is,

$$E = \tfrac{1}{2} w \tan^2\left(45^\circ - \frac{\varphi}{2}\right) \times (33^2 - 8^2) \times 10 = 146{,}580 \text{ lbs.}$$

Otherwise: at N, $p = 228.8$ lbs. sq. ft., at B, $p = 943.8$ lbs. sq. ft.

$$\therefore E = \tfrac{1}{2}(228.8 + 943.8) \times 25 \times 10 = 146{,}580 \text{ lbs.}$$

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[*] Strictly, the value of Q should be computed for beams only 8′ 6″ long.

This thrust acts, Art. 51,

$$c = \left(1 + \frac{h'}{h + 2\,h'}\right)\frac{h}{3} = \left(1 + \frac{8}{25 + 16}\right)\frac{25}{3} = 9.96 \text{ ft.}$$

above B.

Resultant of Weights in Magnitude and Position. The resultants of the weights of concrete and filling, with and without surcharge over IN, must next be ascertained for 10 ft. length of wall, including one counterfort. The filling, to the left of the plane BN, consists of a triangular prism over the counterfort and a rectangular prism between the counterforts. The concrete section is divided up as indicated. Moments were taken about D.

WEIGHTS AND MOMENTS ABOUT *D*

Prism	Length, Feet	Area of Cross-Section, Square Feet	Weight, Pounds	Arm, Feet	Moment, Foot-Pounds
Filling	1.5	22.5 × 5	16,875	+6.67	+112,500
Filling	8.5	22.5 × 10	191,250	+5.	+956,250
Surcharge	10.	8 × 10	80,000	+5.	+400,000
Wall					
DB	10.	2.5 × 10	37,500	+5.	+187,500
Projection	10.	2 × 2	6,000	+9.	+ 54,000
Counterfort	1.5	22.5 × 5	25,312	+3.33	+ 84,373
Stem	10.	25 × 1	37,500	−0.5	− 18,750
Toe	10.	1 × 4.5	6,750	−3.25	− 21,937
Toe	10.	2.5 × 2.25	8,437	−2.5	− 21,094

With surcharge over IN, $W_2 = 409{,}624$, $M_2 = 1{,}732{,}842$.

Without surcharge over IN, $W_1 = 329{,}624$, $M_1 = 1{,}332{,}842$.

The surcharge here replaces the train loads, which may occupy one or several tracks, parallel to the wall. Thus, without ultra refinement as to the exact position of the tracks, the surcharge will be supposed to extend to BN or to I, according as the one or the other position will give the largest stresses at any section considered. Fig. 53.

The distance from D to the line of action of the resultant of the combined weights is:

Case (*a*), without surcharge over $IN = \dfrac{1{,}332{,}842}{329{,}624} = 4.04$ ft., *Case* (*b*), with surcharge over *IN* $= \dfrac{1,732,842}{409,624} = 4.23$ ft.

Thus the resultant in Case (*a*) acts 9.54 ft. from *A*, in Case (*b*) 9.73 ft. from *A*.

In Case (*a*), let R_1 denote the resultant of *E* and W_1;

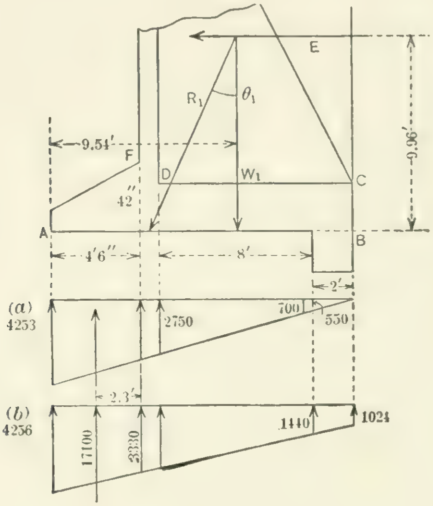

FIG. 53

In Case (*b*), let R_2 denote the resultant of *E* and W_2; also let θ_1 denote the angle R_1 makes with the vertical;

let θ_2 denote the angle R_2 makes with the vertical.

$$\therefore\ \tan\theta_1 = \frac{E}{W_1} = \frac{146,600}{329,624} = 0.445,$$

$$\tan\theta_2 = \frac{E}{W_2} = \frac{146,600}{409,624} = 0.358.$$

From Fig. 53 it is seen that R_1 cuts *AB*, $9.54 - 9.96 \tan\theta_1 = 9.54 - 4.43 = 5.11$ ft., to the right of *A* and R_2 cuts *AB*, $9.73 - 9.96 \tan\theta_2 = 9.73 - 3.56 = 6.17$ ft. to the right of *A* (R_2 was not drawn to avoid confusing the figure).

Resistance to Sliding. Let σ_1 be the factor of safety against sliding in Case (*a*); σ_2 the factor in Case (*b*) $\therefore \sigma_1 E = f'W_1$

$$\therefore \sigma_1 \frac{E}{W_1} = \sigma_1 \tan \theta_1 = f' \therefore \sigma_1 = \frac{f'}{\tan \theta_1} = \frac{0.500}{0.445} = 1.12$$

Similarly in Case (*b*), $\sigma_2 = \dfrac{f'}{\tan \theta_2} = \dfrac{0.500}{0.358} = 1.40.$

The increase in these factors, due to the projection at B, is marked and can be estimated as in the previous example.

Soil Pressures. In Case (*a*), R_1 cuts AB practically at the third point; hence the unit pressure at B is zero and at A double the mean; or since W_1 corresponds to 10 ft. length of wall, the pressure at $A = 2 \times \dfrac{329,624}{15.5 \times 10} = 4253$ lbs. sq. ft.

In Case (*b*), R_2 meets AB, $7.75 - 6.17 = 1.58$ ft. to the left of the center of $AB \therefore l = 15.5$, $a = 1.58$.

$$p = \frac{40,962}{15.5}\left(1 \pm \frac{9.48}{15.5}\right) = 2640\ (1 \pm 0.612)$$

$$= 4256 \text{ lbs. sq. ft. at } A, 1024 \text{ lbs. sq. ft. at } B.$$

These pressures, for both Cases (*a*) and (*b*), are laid off as ordinates in Fig. 53 and the diagrams of soil pressures completed as shown. The unit pressures vertically under F, D, etc., were measured to scale and marked on the figure.

The soil pressure, at any particular distance from A, is the average for the 10 ft. length of wall. Under the toe, the pressure is probably uniform along a line at the same distance throughout from the front edge of the footing, but along such a line of the heel slab the soil pressure is least under the counterforts and greatest midway between counterforts, particularly toward the rear of the heel slab. Reasons will be given for this view when the heel slab design is taken up.

Design of Toe. Take the section at F, whose vertical height is 42″ and breadth $b = 12''$ perpendicular to the plane of the paper. The total upward acting, soil reaction on the toe, for 1 ft. length of wall is greatest for Case (*b*) and $= \frac{1}{2}(4256 + 3330) \times 4.5 = 17,100$ lbs.

Its resultant acts 2.3 ft. to the left of F; hence the bending moment at the section at F is,

$$M = 17{,}100 \times 2.3 \times 12 = 471{,}960 \text{ in.-lbs.}$$

The shear at the section at $F = 17{,}100$ lbs. $= Q$.

The upper surface of the toe makes the angle $\beta = 29^\circ\ 07'$ with the horizontal. For an assumed steel percentage 0.2, from App., Fig. 9, we find $j = 0.92$. Assume the steel reinforcement $3''$ from AB; then $d = 42 - 3 = 39''$. $jd = 35.9$. From App., eq. (14), assuming $f_s = 16{,}000$ lb./in.^{2},

$$A = \frac{M}{f_s . jd} = \frac{471{,}960}{16{,}000 \times 35.9} = 0.822 \text{ sq. in}$$

$\frac{5}{8}''$ □ bars, $5''$ c to c, giving $A = 0.9408$ for $b = 12''$, will be used. Then, $p = A/bd = 0.002$, as assumed.

$$Q - \frac{M \tan \beta}{d} = 17{,}100 - \frac{471{,}960 \times 0.557}{39} = 10{,}360 \text{ lbs.}$$

∴ App., eq. (34),

$$u_1 . jd\, \Sigma o_1 = 10{,}360 = vb . jd$$

$$\text{For } b = 12'',\ \Sigma o_1 = \frac{12}{5}(2.5) = 6.$$

$$\therefore \text{ bond stress } u_1 = \frac{10{,}360}{35.9 \times 6} = 48.2 \text{ lb./in.}^2,$$

$$\text{shear } v = \frac{10{,}360}{12 \times 35.9} = 24 \text{ lb./in.}^2,$$

both safe values. In fact, since the weight of toe and reaction friction on base were omitted in computing Q and M, the true values of A, u_1 and v, are all less than the above.

The $\frac{5}{8}''$ □ bars, $5''$ c to c, will be placed $3''$ above the base AB of the footing, every third bar extending 13.5 ft. from A, the remaining bars 7 ft. from A.

Heel Slab. The heel slab between counterforts up to "projection" is supported on three sides by the face slab and counterforts and to some extent on the rear, by the stiffer beam over and including the projection. Other considerations, too, add to the complexity of a solution. The thrust of the earth against the face slab is first carried by beam action to the counterforts, which rotate slightly under the pulls. The consequent upward movement at the rear tends to lift the entire heel slab. However, the part of this heel slab between counterforts is very much more flexible than the remainder joined to the stiff counterfort, hence it will tend to deflect more. But this increased deflection is at once opposed by an increased soil reaction. Hence if we suppose a beam cut out from the heel slab by planes 1 ft. apart and parallel to the face slab, as shown by the dotted lines Fig. 52 (plan), the soil reaction for such a beam, 10 ft. long, center to center of counterforts, has an average value given by the ordinate under its center of the soil pressure diagram, Fig. 53, but the unit soil reaction is less than the mean, under the counterfort and greater than the mean near the center of the beam. In fact, it is probable, for ordinary designs, that the soil reaction under the "projection" at *B* may be nothing at the counterforts and sufficient at the center to prevent any considerable deflection of the supposed beam. The assumption of a uniform soil pressure over a beam, such as is indicated in Fig. 52 (plan) by the dotted lines, is thus on the side of safety.

As a basis for a practical computation, such beams will be treated as independent, continuous beams, with moments at the middle and ends of $\frac{1}{12} wl^2$, $-\frac{1}{12} wl^2$, respectively. Consider, first, the beam cut from the heel slab next the projection, 1 ft. wide, $l = 10$ ft. long and 30 in. deep.

From Fig. 53, the soil reaction in Case (*a*) is 700 and in Case (*b*) 1550 lbs. sq. ft. In Case (*a*) the weight of earth above the beam is $22.5 \times 1 \times 100 = 2250$ lbs. sq. ft., and in Case (*b*) $30.5 \times 1 \times 100 = 3050$ lbs. sq. ft. The weight of beam per linear foot is $2.5 \times 150 = 375$ lbs. Hence the supposed beam will be subjected to a downward acting load,

in Case (*a*), of $2250 + 375 - 700 = 1925$ lbs. sq. ft.,
in Case (*b*), of $3050 + 375 - 1550 = 1875$ lbs. sq. ft.

The first case gives the greater load, hence substitute $w = 1925$, $l = 10$, in the formula, $M = \frac{1}{12} wl^2 \times 12 = 1925 \times 10^2$

= 192,500 in.-lbs. With $d = 30 - 3 = 27''$, we have by the approximate formula:

$$A = \frac{M}{f_s jd} = \frac{192{,}500}{\frac{7}{8}(16{,}000) \times 27} = 0.510 \text{ sq. in.}$$

Use $\frac{5}{8}''$ □ bars, 8″ c to c, giving $A = 0.588$ sq. in., placed 3″ from the bottom for the whole length and $\frac{5}{8}''$ □ bars, 8″ c to c, placed 3″ from the top at counterforts and extending 2 ft. beyond them on either side.

The shear at the edge of the counterfort is $Q = 1925 \times 4.25 = 8180$ lbs.

There is no need to take the shear at the center of counterforts, since there, $d > 27''$.

The unit shear is, taking $j = \frac{7}{8}$ approximately,

$$v = \frac{Q}{bjd} = \frac{8180}{12 \times \frac{7}{8} \times 27} = 29 \text{ lbs. sq. in.}$$

For the $\frac{5}{8}''$ □, 8″ c to c, $\Sigma o = \frac{12}{8}(2.5) = 3.75$

$$\therefore u = \frac{Q}{jd\, \Sigma o} = \frac{8}{7} \cdot \frac{8180}{27 \times 3.75} = 92.4 \text{ lbs. sq. in.}$$

For similar beams nearer the face wall, the soil reaction is greater and the shear less, and as the $\frac{5}{8}''$ □ bars, 8″ c to c, will be used throughout, such beams are over-reinforced and can assist the extreme one just examined, so that the value $u = 92.4$ lbs. sq. in. is really never exerted. In fact, the heel slab has to act as a whole and is materially assisted too, by its connection with the face wall which was ignored in the computation.

Let us next design the reinforcement for the rearmost 2 ft. width of heel slab, including the projection, Fig. 52. This gives a beam 24″ wide, 54″ deep and 10′ long to examine.

It is easily shown that in Case (*a*), the weight of beam and earth over it, less the soil reaction, is 56,000 lbs. nearly. Hence proceeding as before,

$$M = \frac{1}{12} wl.\, l \times 12 = 56{,}000 \times 10 = 560{,}000 \text{ in.-lbs.}$$

Hence with $d = 54 - 3 = 51''$

$$A = \frac{M}{f_s jd} = \frac{8}{7} \frac{560{,}000}{16{,}000 \times 51} = 0.785 \text{ sq. in.}$$

This is the area of metal in 24″ width.

Use three ⅝″ □ bars, say 7″ c to c $\therefore A = 1.176$ for three bars. The three bars are to be placed, 3″ above the bottom throughout. Other ⅝″ □ bars, 7″ c to c, are to be placed 3″ below the top at the counterforts and will extend 2′ 6″ in each longitudinal direction. The latter will provide for the negative moments.

The shear at edge of counterforts = ½ (8.5 × 5600) = 23,800 lbs. = Q.

$$\text{Unit shear} = v = {}^8/_7 \frac{Q}{bd} = {}^8/_7 \frac{23{,}800}{24 \times 51} = 22.3 \text{ lbs. sq. in.}$$

For bond stress, $\Sigma o = 3 \times 2.5 = 7.5$, for the 3 bars,

$$\therefore u = {}^8/_7 \frac{Q}{d\,\Sigma o} = {}^8/_7 \frac{23{,}800}{51 \times 7.5} = 71.8 \text{ lbs. sq. in.}$$

Both u and v are within safe limits.

Vertical Rods, Tying the Heel Slab to the Counterfort. In the beam 1 ft. wide, 2.5 ft. deep and 10 ft. long, regarded as cut out of the heel slab next the projection, the load on it between counterforts was found above to be 1925 lbs. sq. ft. or 1925 × 8.5 = 16,360 lbs. To this, add the weight of the part of the beam under the counterfort, 1 × 1.5 × 2.5 × 150 = 560 lbs., making a total of 16,920 lbs. as the load to be carried by the vertical rods. This requires an area = 16,920/16,000 = 1.06 sq. ins. in 12″ width.

⅝″ □ rods, 8″ c to c will be placed in pairs (giving A = 1.176 sq. ins.) or rather U rods will be used, passing under the horizontal rods running from the toe, and the ends will extend vertically upward in the counterfort, to, or near to, the inclined rods. As the load on the vertical ties steadily diminishes as the face slab is approached, ⅝″ □ rods were used throughout, but the spacing was varied as indicated in Fig. 54. Thus there are 3 pairs of rods, 8″ c to c, area 2.352, and nearer the face wall, 3 pairs of rods, 16″ c to c, area 2.352, giving a total area of 4.704 sq. ins., that can carry a total stress of 4.704 × 16,000 = 75,000 lbs. No vertical ties are needed within 16″ from the face slab, as it is compressive area, App., Art. 10.

As a check, consider the whole load that must be carried by the vertical ties to the left of the "projection."

Weight of earth over heel slab between counterforts, Case (*a*), $= 8 \times 8.5 \times 22.5 \times 100 = 153{,}000$ lbs.

Weight of slab 10 ft. long $= 8 \times 10 \times 2.5 \times 150 = 30{,}000$ lbs.

Soil reaction, Case (*a*), $= \frac{1}{2}(2750 + 560) \times 8 \times 10 = 132{,}400$ lbs.

Subtracting this from the sum of the first two, we have 51,000 lbs. as the total load to be carried by the vertical rods. They were designed to care for a total stress of 75,000 lbs., which is more than adequate. In fact, part of the load is carried by beam action directly to the face wall, so that the total is really less than 51,000 lbs., as computed above.

Counterfort. The earth thrust on the face wall has been supposed to be carried to the counterforts, which hypothesis is on the side of safety, since part of this thrust below and slightly above F is carried directly to the base.

Assuming, however, that the horizontal earth thrust on 10 ft. length of wall, for the height of the counterfort, 22.5 ft., is all carried by one counterfort, the resultant of this thrust in amount and position is desired.

In amount it is, see Fig. 52,

$$E = \tfrac{1}{2}(230 + 870) \times 22.5 \times 10 = 123{,}750 \text{ lbs.}$$

and it acts above C,

$$c = \left(1 + \frac{h'}{h + 2h'}\right)\frac{h}{3} = \left(1 + \frac{8}{22.5 + 16}\right)\frac{22.5}{3} = 9.06 \text{ ft.}$$

The value of E is found by aid of the pressure diagram Fig. 52 (drawn to a larger scale), where the earth pressures at N and C are found to be 230 and 870 lbs. sq. ft. respectively. Similarly, the unit pressures at $h = 7$, $h = 12$, $h = 17$, are found to be 435, 580, and 720 lbs. sq. ft. as marked on the figure, and the earth thrusts for depths of 7, 12, and 17 ft. are computed, as well as the distances c from the lower edge of the area pressed to the successive resultants, as in the illustration above. The results are put in columns E and c of the following table, and the resulting bending moments (in ft.-lbs.) are then computed and inserted in the column headed $M = Ec$.

The reinforcement of the counterfort, for purposes of computation, will consist of inclined rods parallel to and 3″ from IC, the back of the counterfort. An exact computation, including the vertical rods that tie the base slab to the counterfort, is given in the Appendix, Art. 10 (also see Art 12), but it will generally suffice in practice to ignore the vertical rods, particularly as it is on the side of safety.

In the resulting approximate computation, horizontal sections are taken, extending through the counterfort to the exposed face of the vertical slab. For any section, d = horizontal distance from front face to inclined rods.

By App., eq. (11), $M_s = f_s A_1 j . d \cos \beta_1 = f_s A_1 j . p$, where $p = d \cos \beta_1$ = perpendicular distance from the front face to the inclined rods. The distances p from points on the front face at $h = 7$, 12, 17, 22.5, were measured to scale and recorded in the table. Note that M is in ft.-lbs. and p in feet.

Since β_1 = angle that IC makes with the vertical, $\tan \beta_1 = 10/22.5 = 0.445 \therefore \beta_1 = 23° 58'$. Let us assume a steel percentage = 0.4; then from App. Fig. 9 (dotted lines) we find $j = 0.906$. As usual, take $f_s = 16{,}000 \therefore$ the formula above reduces to,

$$12\,M = 16{,}000 \times 0.906 \times 12\,p . A_1 \therefore A_1 = \frac{M}{14{,}500\,p},$$

where A_1 is in square inches. The areas A_1 are those of right sections of the inclined rods at the distances h ft. below the top of wall.

h, Ft.	E, Lbs.	c, Ft.	$M = Ec$, Ft. Lbs.	p, Ft.	$A_1 = \frac{M}{14500p}$, Sq. In.
7	23,310	3.15	73,400	3.5	1.45
12	48,600	5.14	249,800	5.5	3.14
17	80,750	7.05	569,300	7.6	5.18
22.5	123,750	9.06	1,121,200	9.8	7.90

Use $1\frac{1}{4}''$ □ bars, two extending to the top, two extending to 8 ft. from the top and finally two bars to extend to $h = 13$, the lower sets of bars being extended sufficiently to secure sufficient bond strength. The total steel area of the six bars = 9.45 sq. in.

For $A_1 = 9.45$, the steel ratio $= 9.45 \div (18 \times 128) = 0.0041$, as assumed. The results are all in excess, which justifies the tacit assumption that all rods are in the same inclined plane, whereas some of them are dropped a little below the others, Fig. 54.

The lower ends of the rods pass into the projection and are hooked over the lowest bars in it, giving ample bond stress, since with hooked ends 40 diameters or $40 \times 1\frac{1}{4} = 50''$ is all the embedment needed and $51''$ is supplied.

It is well to test the bond and shear stresses. In doing so, note that the external shear = E, as computed above. Also, the breadth of the counterfort, $b = 18''$, and $\Sigma o = 10$, 20 or 30 sq. ins., according as 2, 4 or 6 bars are used. Assume $j = 0.906$ as before.

By App., eqs. (29) and (33), neglecting the term in M, we have, bond stress $= u_1 = \dfrac{E}{jd\,\Sigma o_1} = \dfrac{E}{0.906\,d\Sigma o_1}$,

unit shear $= v = \dfrac{E}{bjd} = \dfrac{E}{18 \times .906\,d} = \dfrac{E}{16.3 \times d}$,

which are in excess of the true values.

h, Ft.	E, Lbs.	d, In.	u_1 Lb. Sq. In.	v Lb. Sq. In.
7	23,310	46	56	31
12	48,600	72	75	42
17	80,750	100	45	50
22.5	123,750	128	36	59

The bond stresses are within working limits, as indeed are the unit shears, since the cantilever is thoroughly reinforced against diagonal tension, not only by the inclined rods, but also by the horizontal and vertical bars, which act as stirrups. In such cases, shearing stresses of from 60 to 120 lbs. sq. in. are allowed, depending on the amount and character of the reinforcement for diagonal tension.

In some designs, the horizontal and vertical bars of the counterfort are cut off as soon as sufficient embedment is secured to carry the bond stress, but it seems advisable to extend the horizontal bars to connection with the inclined ones, and the vertical bars the full length to the inclined bars, or at least to, say, ⅔ of that length. Such bars act like stirrups in resisting diagonal tension and are very effective in reinforcing the large slab against shrinkage stresses.

Where the inclined bars are bent, the radius of the bend for square bars should not be less than 25 diameters, and for round bars 20 diameters (App., Art. 21), otherwise the concrete will be over-stressed.

Pull of Face Slab on Counterfort. The earth thrust for a width of 8.5 ft. between the counterforts is carried by beam action of the face slab to the counterforts. The corresponding pull of the vertical slab on the counterforts is resisted by the horizontal ties of the counterfort, above alluded to. From the unit pressure diagram of Fig. 52, the total earth thrust on a beam cut out of the vertical slab 1 ft. high, is,

$$\text{at } h = 7, \quad 435 \times 8.5 = 3697 \text{ lbs.,}$$
$$\text{at } h = 22.5, \; 870 \times 8.5 = 7395 \text{ lbs.}$$

The latter thrust will require a metal area = 7395 ÷ 16,000 = 0.462 sq. in., which can be supplied by a pair of ½″ ☐ rods lying in the same horizontal plane. Such pairs are spaced 12″ apart vertically from $h = 7$ to $h = 22$. From the top of the wall to $h = 7$, ⅜″ ☐ rods, 14″ apart vertically will suffice.

The bond stress developed in the 12″ thickness of face slab is inadequate, since even with hooked ends the length of embedment should be 40 × ½ = 20″, whereas only 10″ is available. It is plain that some additional means of resisting the pull must be provided. One simple way is to pass the horizontal rods of the counterfort around or through vertical steel angles placed near the exposed face of the vertical slab. The pairs of rods spoken of thus become continuous U rods.

The horizontal bars of the face slab can either pass through holes punched in these angles or they may lie entirely outside the angles. The effective bearing area of the angles should be at least 25 times the area of the rods, App., Art. 21.

Reinforcement against shrinkage and temperature stresses in the vertical slab. To reinforce against shrinkage and for spacing, vertical ½″ □ bars, 24″ apart, will be placed near the front and rear faces.

The area of the cross-section of the vertical slab for a depth of 22 ft., is $22 \times 12 \times 12 = 3168$ sq. in. The metal area required for temperature reinforcement is ⅓ of 1% = 10.56 sq. in.

To avoid, as much as possible, a multiplicity of sizes, change the 48 horizontal ½″ □ bars, designed for beam action, to ⅝″ □ bars, similarly spaced, placed near the front face and add 48 horizontal ⅜″ □ bars, similarly spaced, near the rear face, as shown in Fig. 54. The total steel area is $48 \times 0.392 + 48 \times 0.141 = 25.584$ sq. in., which is greater than the total area of the original 48 ½″ □ bars (= 12 sq. in.) + 10.56 sq. in. required for temperatures stresses.

The area thus added for temperature stresses will be found to be ample for $h = 7$ to $h = 22$ and sufficient from $h = 0$ to $h = 7$, considering that the steel area designed for beam action over this upper portion is considerably in excess.

The ⅜″ □ bars should be wired at intervals to the ½″ □ bars, 5 ft. each in length, required near the rear face to resist the negative moments at the counterforts.

See Fig. 54, for the final design.

The three *types* of gravity, T and counterforted walls hitherto discussed, are standard, though of course the ratio of length of toe to heel varies to suit special conditions. Thus it often happens, on account of a limited right of way, as in track elevation in cities, that the toe of a retaining wall must be made very short or else done away with altogether, the resulting type being an L wall. Some interesting types of walls without toes.

are given in *Engineering News* for April 20, 1911, and the issue for July 28, 1910, shows a great variety of types of gravity walls as actually constructed

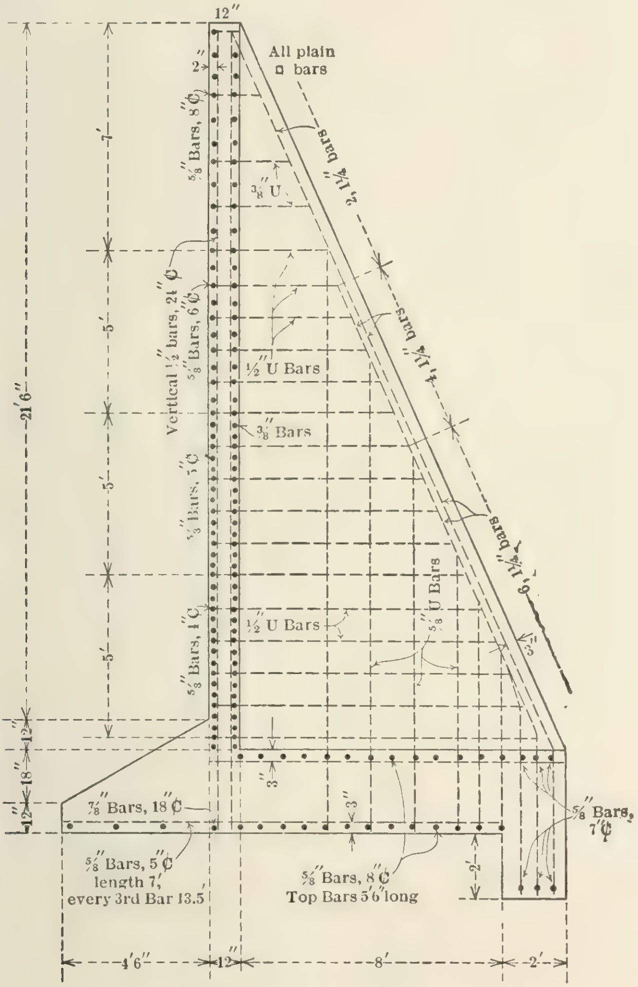

FIG. 54

In Fig. 55* is shown a section of one type of retaining wall, used on the Chicago Track-Elevation work, Rock Island lines.

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* Given in *Engineering News*, April 8, 1915.

The walls are built in sections 35 ft. long to allow for temperature changes, vertical keys being used at the expansion joints.

The usual weep-holes are omitted, but along the back of the wall are laid inclined drains of 6-in. porous tile, on a grade of 0.5% extending from subgrade level ultimately to an 8-in. pipe through the wall, having connection with the catchbasin of a city sewer. The duct at the top is for electric wires, cables and telegraph lines. The back of the wall is waterproofed with a tar-pitch composition, a strip of burlap and felt being placed over each expansion joint and well mopped with the composition. It is highly desirable to keep water out of the expansion joints, since its freezing is disintegrating to the concrete.

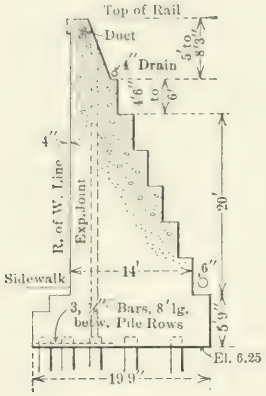

FIG. 55

The reader will notice the usual stepped wall, which is always advisable. Sometimes, in cold climates, the inner face of a retaining wall for 3 or 4 ft. down, is battered at 45° or less, to allow for the freezing of the earth and its consequent expansion.

CHAPTER V

EARTH ENDOWED WITH BOTH COHESION AND FRICTION

89. THE theory will now be developed of the pressures in an unlimited mass of earth endowed with cohesion as well as friction. The old hypothesis of Coulomb will be used, that the friction on a plane in the interior of a mass of earth, is equal to the normal pressure multiplied by the coefficient of friction, $f = \tan \varphi$, and the cohesion is equal to the area of the plane considered multiplied by c, the cohesion per unit of area. In other words, the friction is proportional to the normal pressure on the area considered, whereas the cohesion is simply proportional to this area and is independent of the normal pressure.

In applying the following theory, it is essential that both φ and c, be determined by experiment, one method for experimenting having been suggested in Art. 3. It would be erroneous to assume for coherent earth, that φ could be found by observing the natural slope. The inclination to the horizontal of the greatest slope at which the earth will stand, is always greater than the true angle of friction φ for coherent earth. This is a consequence of eq. 1, Art. 3 and it will be incidentally shown in Art. 90, where the greatest possible surface slope is found that corresponds to a given angle of internal friction φ and a given depth.

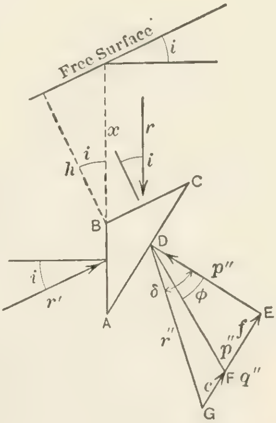

FIG. 56

In Fig. 56, ABC represents a wedge of rupture at a depth x below the free surface, whose inclination to the horizontal is i. The wedge is supposed to have a length unity perpendicular to the plane of the paper, the lengths BC, AB and AC, being taken so small, that the unit stresses r, r', r''

upon the faces corresponding, can be regarded as uniform. As in the Rankine theory for non-coherent earth, r', the unit stress on the vertical plane AB acts parallel to the top slope and the vertical unit stress r, conjugate to r', acts on a plane BC parallel to the top slope or free surface. The stresses r and r', thus have a common obliquity i.

The reaction of the earth below the plane of rupture AC on that plane is indicated for the case of *active thrust*, referring to incipient motion down the plane. In this case, the normal unit stress being p'', the tangential component, acting up along AC, equals $q'' = c + p''f = c + p'' \tan \varphi$, by Coulomb's law quoted.

The frictional unit resistance being $FE = p'' \tan \varphi$, it follows that $EDF = \varphi$. The unit cohesive resistance $= GF = c$, so that the total tangential unit stress $q'' = GE = c + p'' \tan \varphi$ and the resultant unit reaction $= GD = r''$. The obliquity of stress on the plane AC will be designated by δ; $\therefore \delta =$ angle EDG. We have,

$$\tan \delta = \frac{GE}{DE} = \frac{q''}{p''} = \frac{c + p'' \tan \varphi}{p''} = \tan \varphi + \frac{c}{p''}.$$

The obliquity δ thus increases as p'' diminishes and is not invariable as for non-coherent earth, where $\delta = \varphi$.

The wedge of rupture with the unit stresses acting on its faces, as drawn here, corresponds to Case (*b*), Fig. 31, when c is made zero. Let the student draw figures for the other three cases (*a*), (*c*) and (*d*), only including the cohesion; remembering that in the case of passive thrust, (*c*) and (*d*), the external force r' is on the point of moving the wedge ABC in the direction A to C and $q'' = c + p'' \tan \varphi$ now acts in direction C to A or opposed to the motion.

The circular diagram of stress is shown in Fig. 57. The ordinate $OL \perp KOJ$, is laid off equal to c and the line KLS is drawn through L, making the angle φ with the line KOJ; its equation being, when p'' and q'' are the variables,

$$q'' = c + p'' \tan \varphi$$

p'' and $\therefore q''$, varies with the depth x.

We have from the figure, $KO = c \cot \varphi$. To draw the circle, for the case of *active* pressure, lay off $OR = r = wx \cos i = wh$, making the angle $ROJ = i$ with OJ and then find by trial, the center P on OJ, so that the circle, with center P and the *least* radius, shall pass through R and be tangent to KLS. Let the tangent point be written S. Then if $p'' = ON$, we have, $q'' = c + p'' \tan \varphi = NS$ and $\tan NOS = \tan \delta = q''/p''$, as should be for the plane of rupture AC of Fig. 56.

Now IS, Fig. 57, corresponds to this plane of rupture AC, Fig. 56, since for any other plane through I as IR, the tangential component MR would be less than the corresponding ordinate q'' to KS (produced if necessary), so that slipping could not be impending on the plane corresponding to IR. Slipping is thus impending only on the plane corresponding to IS, so that it is one plane of rupture. We cannot determine the point S as in Art. 62, since the obliquity δ varies with p'' or with x $\left(\text{for, } \tan \delta = \dfrac{c + p'' \tan \varphi}{p''}\right)$ and in fact, δ is unknown until ascertained by this construction. The circle drawn is the only one that will satisfy the equations above for the plane of rupture.*

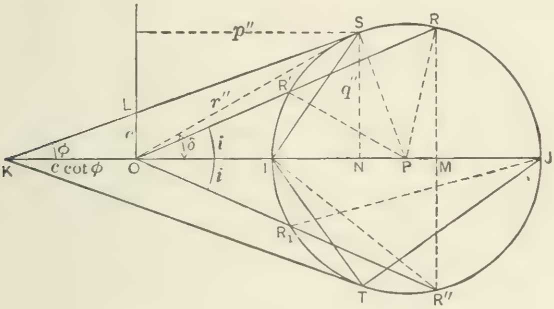

FIG. 57

The relation between Fig. 57 and the ellipse of stress Fig. 29, should be kept clearly in mind. Thus, if OR, Fig. 57, cuts the circle in R', then if we assume

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* This diagram is similar to the one given by Prof. O. H. Basquin for coherent earth, in his paper on "The Circular Diagram of Stress and its Application to the Theory of Internal Friction," Western Soc. of Engs., Sept. 9, 1912.

$$OP = OP' \text{ of Fig. 29} = OP \text{ of Fig. 57}$$
$$PR = P'R' \text{ of Fig. 29} = PR = PR' \text{ of Fig. 57,}$$

also that the obliquity, $ROP = R'OP'$, is the same in both figures, then it follows that OR in one figure $= OR$ in the other, and similarly for OR'. Hence the lengths OR, OR', of Fig. 57 represent the conjugate stresses OR, OR' of Fig. 29, having the common obliquity ROP. Also OI and OJ give the lengths of the semi-axes of the ellipse of stress.

If the student will now re-read Arts. 63-64 and the part of Art. 65 not involving the derivation of (41) and (42), he will find all the conclusions to hold on simply replacing the φ of Fig. 29 by the corresponding δ of Fig. 57. The latter figure is thus the true circular diagram of stress.*

To find the *active* conjugate thrust r' for a given vertical depth x, we compute $r = wx \cos i = wh$ and having laid off the angle $ROJ = i$ and $OR = wh$ (to the scale of force) we draw the circle of *least* radius that passes through R, touches KLS and has its center on KOJ. If this circle cuts OR at R', then OR', to the scale of force $= r'$, Art. 63.

To find the *passive* conjugate stress r' for the given vertical depth x, or the normal depth h, lay off, to the scale of force, $r = wh = OR'$ and draw the circle of *largest* radius through R', tangent to KLS and with its center on KOJ. Then if OR' produced cuts the circle at R, the conjugate passive thrust $= r' = OR$, to the scale of force. Certain conclusions of Art. 65 likewise apply to the angles β, γ, β', γ' (having the relative positions shown in Fig. 36) that the planes of rupture make with the vertical. Thus in Fig. 57, $\beta = \frac{1}{2}\, SPR'$, $\gamma = SPI - \beta = 90° - \varphi - \beta$. $\beta' = \frac{1}{2}\, SPR$, $\gamma' = SPJ - \beta' = 90° + \varphi - \beta'$. Compare with Fig. 36, noting that SOP is now $= \delta$. When $c = 0$, then $\delta = \varphi$ and the construction of Fig. 37 for non-coherent earth applies. For $i \leq \varphi$ and x very large, c/OR is very small and the results approach those for $c = 0$.

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* It will prove instructive to the student to draw the ellipse of stress in true position, after the method outlined in "Note," Art. 63.

Example. Let $c = 100$ lbs. per sq. ft., $w = 100$ lbs. per cu. ft., $\varphi = i = 30°$, find by the construction for active thrust, the values of the angles γ, made by the plane of rupture IS with the vertical R_1I and the values of the conjugate stress r' for the values of x given.*

x Ft.	h Ft.	γ	r' Lb. per Sq. Ft.	β
3.46	3.00	30°	0	30°
5.26	4.55	35°	84	25°
8.42	7.29	40°	253	20°
15.00	13.00	45°	646	15°
32.70	28.30	50°	1,847	10°
124.00	107.30	55°	8,738	5°

If the values of x are laid off along a vertical axis and the values of r' plotted as horizontal ordinates, it will be found that the line joining the extremities of the r''s is very slightly concave upwards. The stress is thus not uniformly increasing and the center of pressure is very slightly below the corresponding center for a stress uniformly increasing from the depth $x = 3.46$ ft., where $r' = 0$ to the full depth considered.

If a line be drawn parallel to the surface, at a depth 3.46 ft. below it (where $r' = 0$) and this be regarded as a new surface for non-coherent earth, the unit thrust for $x = 124$ ft. or for the depth $(124 - 3.5)$ referred to the new surface for non-coherent earth is, by the usual formula for such earth, $120.5 \cos \varphi = 10{,}440$ lbs. per sq. ft. This differs materially from the true thrust 8738 lbs. found above. See a corresponding comparison in Art. 96, for earth surface horizontal.

It is seen from the table that γ increases with x, hence the surface of rupture considered makes an angle with the vertical which increases with the depth. Hence the entire surface of rupture from $x = 3.46$ ft. down, is concave upwards. For $x = 3.46$, $r' = 0$ or the points O and I coincide. For $x < 3.46$ ft., it will be shown in Art. 92, for a stable unlimited mass of earth, that the conjugate stress $r' = 0$.

90. Maximum Surface Slope for a Given Depth.

$$\varphi < i < 45° + \frac{\varphi}{2}.$$

It is evident from Fig. 57, for *active* thrust, where $r' < r$, since the circle drawn, passing through R, center on OJ and touching KLS, is the one with the *smallest* radius, that OR cannot lie to the left of OS; hence for active thrust, the greatest value, for a given depth x, that i can attain is δ, when OR coincides with OS.* A simple construction will give this maximum value of i, when c, φ and the normal depth $h = x \cos i$ are given. Thus in Fig. 58, having laid off $OL = c$, where OL is $\perp KJ$, and drawn the line KLS making the angle φ with KJ; then since $i = \delta$, we have, $r = wx \cos i = wh = OS$. Hence, with O as center, wh (to scale) as radius, describe an arc, cutting KL produced at S; whence $\angle SOJ =$ maximum value of i for the normal depth h assumed. After i has been found by this construction, then $x = h \sec i$, can be computed, but x cannot be assumed in the first instance.

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* These values were in reality computed from formulas, taking γ for the independent variable. They were checked by the diagram.

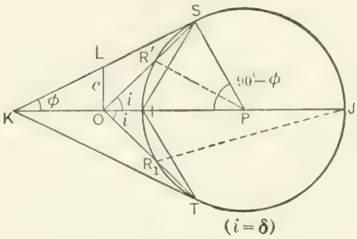

FIG. 58

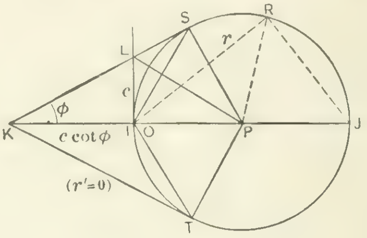

FIG. 59

As a special case, when the point O coincides with I, Fig. 59, or for a normal depth h such that $wh = OS = IS$, then since $KPS = 90° - \varphi$ and from the isosceles triangle OPS, $2\,SOP = 180° - OPS = 90° + \varphi$ $\therefore$ the maximum value of $i = SOP$

$$= SIJ = 45° + \frac{\varphi}{2}.$$

At this depth, the conjugate stress $r' = OR' = 0$. See Art. 92 for the corresponding value of x. As the vertical depth increases beyond this, the length OS, Fig. 58, increases, hence the maximum value of $i = SOJ$ diminishes and approaches the angle φ for great depths, since then $\angle OSL$ is very small. For coherent earth, maximum $i = SOJ$, Fig. 58, is always greater than φ.

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* For *passive* thrust, where the *largest* radius is used to describe a circle with center on OJ, passing through R' and touching KLS, $OR'R$ can lie above OS and it is possible to have $i > \delta$. This can be visualized, if in Fig. 57, O is taken near I, so that $OR'R$ lies above OS. In this case, $r' = OR$, can be less than, equal to or greater than $r = OR'$.

Having found by the preceding construction the greatest possible surface slope for a given h, if for this i, h is further increased, slipping will occur at the new depth, unless the earth at that depth is confined by walls, natural or otherwise, which prevent motion and whose resistance thus introduces extraneous forces not contemplated in the theory of the homogeneous mass of earth of indefinite extent, subjected to no external force but its own weight.

For the limiting case above, $i = \delta$ or when the surface attains its greatest possible slope for the given depth, the construction, Fig. 58, gives the inclination of the planes of rupture to the vertical, at the given depth.

As proved in Art. 65, when the figure 58 is supposed to be revolved about I until the chord IR_1 is vertical (or R_1J horizontal), then for active thrust, the planes of rupture IT and IS will have their true directions. The plane IT makes the angle R_1IT with the vertical. This angle $= R'IS$, is measured by ½ arc $R'S$, which is also the measure of the angle $OSK = i - \varphi$; whence $\beta = R_1IT = i - \varphi$. By Art. 65, this corresponds to case (a).

For case (b), shown in Fig. 56, the plane of rupture corresponding to AC, is IS in Fig. 58, which makes an angle with the horizontal R_1J which is measured by $\frac{1}{2}(SJ - R_1I) = \frac{1}{2}(SJ - R'I)$, which is the measure of $\delta = i$. Hence the second plane of rupture is parallel to the free surface. See Art. 94, Fig. 61, for application of the conclusions above.

91. The Greatest Depth at which Equilibrium is Possible when $i > \varphi$. When the surface slopes indefinitely, its angle with the horizontal being i, it has been seen that for $i = \delta$, slipping is impending on the plane AC, Fig. 56 where $r = wh = OS$, Fig. 58.

From the triangle OLS, Fig. 58, we have,

$$\frac{r}{c} = \frac{OS}{OL} = \frac{\sin OLS}{\sin OSL} = \frac{\sin(90^\circ + \varphi)}{\sin(i - \varphi)} = \frac{\cos\varphi}{\sin(i - \varphi)}.$$

Letting x_o be the corresponding vertical depth $\therefore$ $r = w\, x_o \cos i$,

$$\therefore x_o = \frac{r}{w \cos i} = \frac{c}{w} \frac{\cos \varphi}{\cos i \sin (i - \varphi)}, \quad i > \varphi.$$

When the surface slopes indefinitely as assumed, equilibrium is impossible at a greater depth than x_o for given values of i and φ.

When i reaches the value $45° + \frac{\varphi}{2}$, the formula reduces to

$x_o = \frac{2\, c}{w} \tan \left(45° + \frac{\varphi}{2}\right)$, which is in agreement with a result above, when the points O and I coincide. See Art. 92.

When $i < \varphi$, it is seen from Fig. 57, that we cannot have $i = \delta$; in this case there can be equilibrium at any depth.

To derive a formula for the active conjugate stress, $r' = OR'$, corresponding to the depth x_o, or to $r = OS = \frac{c \cos \varphi}{\sin (i - \varphi)}$, note in Fig. 58, that,

$$OPR' = IPS - R'PS = IPS - 2 . OSK =$$
$$(90° - \varphi) - 2\,(i - \varphi) = 90° - (2\, i - \varphi)$$

Now apply the law of sines to triangles IPS and $OR'P$ in turn and substitute the above value of r.

$$PS = OS \frac{\sin i}{\cos \varphi} = \frac{c \sin i}{\sin (i - \varphi)},$$

$$r_o' = OR' = PR' \frac{\cos (2\, i - \varphi)}{\sin i} = \frac{c \cos (2\, i - \varphi)}{\sin (i - \varphi)},$$

where r_o' is the active conjugate stress at the depth x_o.

As seen in Art. 90, at this depth,

$$\beta = i - \varphi, \ \gamma = 90° - i.$$

92. **The vertical depth x′, at which the conjugate stress r′ is zero,** can be easily found. In this case, O coincides with I,

since then $r' = OR' = o$, $ROJ = i$ and in Fig. 59, since $KPS = 90^\circ - \varphi$, we have $OPL = 45^\circ - \frac{\varphi}{2}$; whence $OP = OL\ cot \left(45^\circ - \frac{\varphi}{2}\right) = c\ tan \left(45^\circ + \frac{\varphi}{2}\right)$. Noting that $ORJ = 90^\circ$, $OR = r = 2\,OP\ cos\ i = 2\,c\ cos\ i\ tan\left(45^\circ + \frac{\varphi}{2}\right)$.

Also, since $r = wx'\ cos\ i$,

$$\therefore\ x' = \frac{2\,c}{w}\ tan\ \left(45^\circ + \frac{\varphi}{2}\right)$$

This important formula gives the vertical depth x' at which there is no active thrust. It is true for any i not greater than $\delta = SOJ = 45^\circ + \frac{\varphi}{2}$.

From Fig. 59, it is seen that when $r' = OR' = OR_1 = o$, R_1 coincides with O; hence the horizontal R_1J or OJ is now in true position. $\angle\ SOL = \angle\ LPO = 45^\circ - \frac{\varphi}{2}$ and the planes of rupture OS and OT make angles of $\left(45^\circ - \frac{\varphi}{2}\right)$ with the vertical.

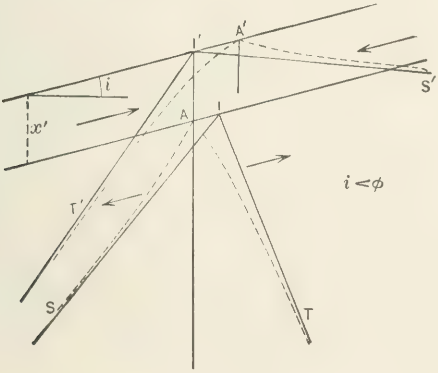

FIG. 60

93. Typical forms of the surfaces of rupture, when $i < \varphi$, are shown in Fig. 60. The curves were sketched for $\varphi = 30^\circ$,

$i = 15°$, $c = 100$ lbs. sq. ft., $w = 100$ lbs. cu. ft. From Art. 92, we have $x' = \frac{2c}{w} \tan\left(45° - \frac{\varphi}{2}\right) = 2 \tan 60° = 3.46$ ft.

For *active* thrust, the surfaces of rupture begin at the depth x', say at A, where they make angles of $\left(45° - \frac{\varphi}{2}\right) = 30°$ on either side of the vertical, Art. 92. The inclination for any depth is readily found from the circular diagram of stress, so that the (dotted) curves of rupture can be approximately drawn.*

h, Ft.	x, Ft.	β	γ	r' Lbs. per Sq. Ft.
3.33	3.46	30.0°	30.0°	0
10	10.35	25.6	34.4	250
20	20.70	23.9	36.1	640
30	31.05	23.3	36.7	1,020
40	41.40	23.0	37.0	1,400
50	51.75	22.9	37.1	1,790
500	517.50	22.0	38.0	19,200

It will be noticed that r' varies, very nearly, according to a linear law.

One way to do this is to draw lines parallel to the free surface at the normal depths h at which the inclinations of the surface of rupture have been found. At A draw a line inclined 30° to the vertical. At some point on this tangent, corresponding to a normal depth intermediate between the first and second normal depths, draw a line having the inclination of the plane of rupture, corresponding to the second normal depth and continue it to and beyond intersection with the line drawn parallel to the surface at this second normal depth. It represents the tangent to the curve at the intersection point.

Continue thus until a sufficient number of enveloping tangents are drawn to sketch the curve. Where the successive tangents make angles of only 2° or 3° with each other, the results are close approximations. As hitherto pointed out, for great depths, the slope of the curve AS or AT approaches that of the surface of rupture IS or IT for non-coherent earth.

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* In the table following, the values of β, $\gamma = (90° - \varphi) - \beta = (60° - \beta)$ and the active conjugate thrust r', are given for various values of h (assumed) and $x = h \sec i = 1.035\, h$, taking $c = 100$ lb./ft.^{2}, $w = 100$ lb./ft.^{3}, $\varphi = 30°$, $i = 15°$.

Similarly, the curves for *passive* resistance (to be discussed later) corresponding to motion up the plane $A'T'$, $A'S'$, at considerable depths, approach in inclination the surfaces of rupture $I'T'$ and $I'S'$ for non-coherent earth.

It will be observed that the surfaces of rupture AS and AT are nearly plane for $i = 15°$, $\varphi = 30°$, assumed. As i increases, the curvature becomes more pronounced. When $i = \varphi$, it has been previously pointed out, Art. 61, that IS is parallel to the free surface and IT is vertical. Hence for $i = \varphi$ at great depths, the curve AS approaches parallelism to the surface and AT approaches the vertical.

Similarly, when $i = \varphi$, $I'S'$ is parallel to the free surface and $I'T'$ becomes vertical. Hence $A'S'$, as x increases indefinitely, approaches parallelism to the free surface and $A'T'$ approaches a vertical.

Recurring to the numerical example of Art. 89, it may be of some interest to know that the equation of the curve of rupture, AS, for active thrust, was found to be, very nearly,*

$$y = 0.2936\, x^{1.63}$$

It must be borne in mind that this equation only refers to the special values, $i = \varphi = 30°$, $c = w = 100$, the greatest depth entering into the computation being 124 ft. However, the curve approaches parallelism to the free surface, as x increases indefinitely, as should be the case. Here y is measured along the surface, say from A' to the left, and x, as hitherto, is measured vertically downward. The curve is, of course, limited to values of $x \geq x'$.

Referring again to Fig. 60, it is well to recall that, in the unlimited mass of earth supposed, the conjugate thrusts on a vertical plane act parallel to the free surface. To appreciate the significance of the curves, it may be noted, relative to $A'T'$, that if an external force acts parallel to the surface and upward, sufficiently great to cause motion, the mass to the left of $A'T'$ will move up along the surface of rupture $T'A'$. If the mass above $A'S'$ is subjected to a force acting down and parallel to the free surface, sufficiently great to cause motion, the mass will move up along the surface of rupture $S'A'$. For the active thrusts, if the mass of earth to the left of AS gives way from any cause, it will descend along the surface of rupture AS, causing, too, a break above A. Lastly, if the earth to the right of AT gives way, it will descend along the surface of rupture AT, the break above A following as a consequence.

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* The derivation of this equation, which involves a modification of a well-known application of the method of least squares, is omitted, since it is mainly of academic interest.

94. Similar remarks apply to Fig. 61, where i is greater than φ but less than $45^\circ + \frac{\varphi}{2}$, Art. 90. Here $A'T'$, $A'S'$, are the curves of rupture for passive thrust, Art. 89, and AS and AT are the curves for active thrust. To sketch the latter, we first lay off vertically downward from the free surface,

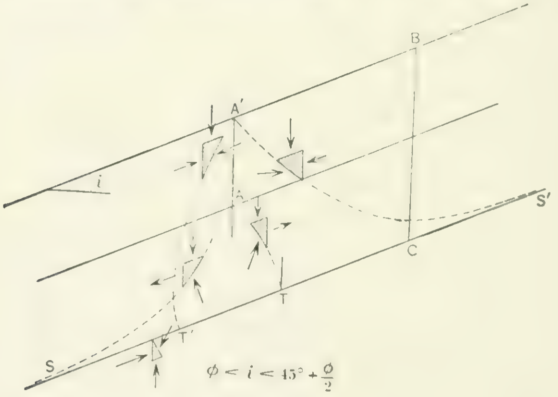

FIG. 61

$$A'A = \frac{2\,c}{w}\,tan\left(45^\circ + \frac{\varphi}{2}\right),\ BC = \frac{c}{w}\,\frac{cos\ \varphi}{cos\ i\ sin\ (i - \varphi)},$$

Arts. 91, 92, and through the point C, draw a parallel $S'C\ TT'$ to the surface. At A, the curves AS and AT make angles of $\left(45^\circ - \frac{\varphi}{2}\right)$, on either side of the vertical, with the latter, Art. 92.

The curve AS is concave upward and approaches CT produced as an asymptote, Art. 90. The curve AT is concave downward and at T makes the angle $(i - \varphi)$ with the vertical. Art. 90.[*] The general shape of the curves can now be sketched.[†]

h, Ft.	x, Ft.	r', Lb. per Sq. Ft.	β	γ
5.18	5.71	0	35° 00′	35° 00′
10.00	11.03	370	24° 45′	45° 15′
15.00	16.54	870	17° 15′	52° 45′
20.00	22.06	1,550	10° 15′	59° 45′
21.60	23.80	1,990	5° 00′	65° 00′

It is seen that r' does not vary uniformly with x, the rate increasing as x increases. This might have been anticipated because the surface of rupture AS is so far removed from a straight line. Regarding r' as varying uniformly between the points given, the total active earth thrust, acting parallel to the surface, is found to be 14,150 pounds. Its resultant acts below the third point.

It has been proved, Arts. 91–92, that at A the active conjugate stress $r' = 0$ and at T or S, $r' = \dfrac{c \cos (2i - \varphi)}{\sin (i - \varphi)}$. For a greater depth than BC, equilibrium is impossible, if the earth is free to move along CT. It would seem, then, if a deep cut was made in a hillside of homogeneous earth below the line CT, that sliding would occur along such a surface as AS.

When $i < 45° + \dfrac{\varphi}{2}$, it will be found that $AA' < BC$, but when i attains its limit, $45° + \dfrac{\varphi}{2}$, Art. 90, then $AA' = BC$, and equilibrium is not possible at a greater depth than AA', unless some external forces are introduced, not considered in the theory above.*

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[*] The table gives the values of r' (active thrust), β and γ, corresponding to various values of h (or, $x = h \sec. i$), when $c = 100$ lb./ft.^{2}, $w = 100$ lb./ft.^{3}, $\varphi = 20°$ and $i = 25°$.

[†] Figs. 60, 61, are similar to those given by Résal in *Poussée des Terres*, II., pp. 34 and 38.

95. Passive Thrust. In Fig. 57 the vertical unit stress on a plane parallel to the surface, at the depth x, $r = wx \cos i = wh$, is now represented by OR' and the conjugate passive unit thrust by OR. The angles with the vertical made by the planes of rupture are,

$$\beta' = \tfrac{1}{2}\, SPR,\ \gamma' = SPJ - \beta' = 90^\circ + \varphi - \beta'.$$

At the surface where $r = OR' = o$, O is at I, Fig. 59, and as proved in Art. 92,

$$r' = 2\, c \cos i \tan\left(45^\circ + \frac{\varphi}{2}\right),$$

$$\beta' = \tfrac{1}{2}\, SPR = SIR = 45^\circ + \frac{\varphi}{2} - i,$$

$$\gamma' = SPJ - \beta' = 90 + \varphi - \beta' = 45^\circ + \frac{\varphi}{2} \times i.$$

These are the angles made by the curves A′T′, A′S′, Fig. 61, with the vertical at the point A'.†

When $i > \varphi$ and $\delta = i$, Fig. 58, since now $r = wh = OR'$, $r' = OS$, we have from the results of Art. 91,

$$r' = OS = \frac{c \cos \varphi}{\sin (i - \varphi)},$$

$$r = OR' = \frac{c \cos (2\, i - \varphi)}{\sin\ (i - \varphi)} = wx \cos i,$$

or r and r' for active thrust are interchanged for passive thrust.

Hence at the depth x given by the last formula $\beta' = \tfrac{1}{2}\, SPR = o$, $\gamma' = SPJ = 90^\circ + \varphi$; $\therefore$ at this depth, the curve $A'T'$, Fig. 61, is vertical and the curve $A'S'$ is inclined at the angle $90^\circ + \varphi$ with the vertical.

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* If in Fig. 61, we take a plane rock surface parallel to the earth surface, at the vertical depth x, the unit pressure on this plane is $r = wx \cos i =$ weight of vertical prism of earth resting on unit area. The components of r parallel and normal to the plane are $r \sin i$, $r \cos i$, respectively. For stable equilibrium, we must have, assuming c and $\tan \varphi$ the same for earth on rock as for earth on earth,

$$r \sin i \leq r \cos i \tan \varphi + c,$$

or,

$$x \leq \frac{c \cos \varphi}{w \cos i \sin (i - \varphi)}$$

The right member is the value of BC, previously found, so that it is again proved, that for a depth greater than BC equilibrium is impossible. The result refers to a uniform slope of earth surface and rock surface of indefinite extent, never exactly realized in practice.

† The horizontal coordinates of curve $A'S'$ are very much reduced to keep the figure within bounds. The true curve lies above the one shown and extends much farther to the right.

When $i > \delta$, as noted in Art. 90, $OR'R$ lies above OS (consider Fig. 57, when o is near I) and β' becomes negative. With R at S, R' is on the arc $AR'S$ below S. As R' moves up along the arc, R moves down. At one point $OR = OR'$ is tangent to the arc IS; finally when R' reaches S, R has the position first occupied by R'. It is assumed that the proper circle is drawn for each intermediate position.

When R' is at S, $r = wh = wx \cos i = OS$, or at the depth $x = x_0$, Art. 91, the conjugate passive thrust $r' = OR = r_0{}'$ = the active conjugate thrust, the computation of Art. 91, exactly applying, considering that for passive thrust $r = OR' = OS$, $r' = OR$, so that R and R' are interchanged. We have also $\beta' = -(i - \varphi)$, $\gamma = 90 + i$.

Since the active and passive thrusts are equal at the depth x_0 the equilibrium there is unstable.

Example. Let $i = 25°$, $\varphi = 20°$, $c = w = 100$. From the results above, the passive thrust at the surface $= 2\,c \cos 25° \tan 55° = 259$ lbs. per sq. ft. and $\beta' = 30°$, $\gamma' = 80°$. At the normal depth, $h = x_0 \cos i = \dfrac{\cos \varphi}{\sin (i - \varphi)} \doteq \dfrac{\cos 20°}{\sin 5°} = 10.8$ ft., the passive thrust = active thrust $= \dfrac{c \cos (2\,i - \varphi)}{\sin (i - \varphi)} =$ 994 lbs. per sq. ft., $\beta' = -(i - \varphi) = -5°$, $\gamma' = 90 + i = 115°$.

At the normal depth, $h = \dfrac{\cos (2\,i - \varphi)}{\sin (i - \varphi)} = 9.94$ ft., the passive thrust $= \dfrac{c \cos \varphi}{\sin (i - \varphi)} = 1080$ lbs. per sq. ft. and $\beta' = 0$, $\gamma' = 90° + \varphi = 110°$. For $h = 5$ ft., prove by constructing the circular diagram that the passive thrust $r' = 810$ lbs. per sq. ft., $\beta' = 11°$, $\gamma' = 99°$. The unit passive thrust increases from the surface down to the normal depth 9.94 ft. and then decreases down to $h = 10.8$ ft.

There is a passive thrust when $i > 45° + \dfrac{\varphi}{2}$, but it is of small interest, since then, there is no active thrust, Art. 90, hence it will simply be stated, without proof, that the curves $A'T'$, $A'S'$, then extend only to the depth, $x' = AA'$, when tension is excluded.

In connection with Fig. 61, let $A'B$ represent a mountain slope, and suppose a break in the upper part, due to excessive rains, say; then if the resulting pressure is sufficient to overcome the passive resistance of the part below, the upper part will slide along a surface similar to $A'AS$ and the lower part along a surface similar to $S'A'$. If the earth below the slide is of a much firmer consistency than that above, the moving earth will flow downward over A' and along the free surface.

96. Active Thrust. Surface Horizontal. See Fig. 62, $i = 0$. In this case, $r = OR = OJ = a = wx$ and the conjugate thrust

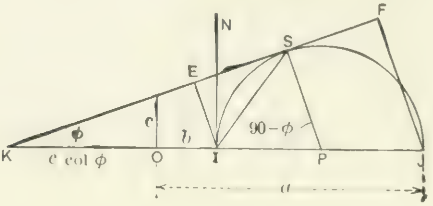

FIG. 62

$r' = OI = b$. On dropping the perpendiculars IE and JF upon the tangent KS, we find $IE = (b + c \cot \varphi) \sin \varphi$, $JF = (a + c \cot \varphi) \sin \varphi$

$$\therefore\ 2\,PS = (a + c \cot \varphi + b + c \cot \varphi) \sin \varphi = a - b.$$

$$\therefore\ b = a\,\frac{1 - \sin \varphi}{1 + \sin \varphi} - 2\,c\,\frac{\sin (90^\circ - \varphi)}{1 + \cos (90^\circ - \varphi)}$$

$$= a \tan^2 \left(45^\circ - \frac{\varphi}{2}\right) - 2\,c\,\frac{2 \sin \left(45^\circ - \frac{\varphi}{2}\right) \cos \left(45^\circ - \frac{\varphi}{2}\right)}{2 \cos^2 \left(45^\circ - \frac{\varphi}{2}\right)}$$

$$\therefore\ b = a \tan^2 \left(45^\circ - \frac{\varphi}{2}\right) - 2\,c \tan \left(45^\circ - \frac{\varphi}{2}\right)$$

$$\therefore\ b = \tan \left(45^\circ - \frac{\varphi}{2}\right) \left[wx \tan \left(45^\circ - \frac{\varphi}{2}\right) - 2\,c\right]$$

Here x is the vertical depth, below the free surface, at which the horizontal unit thrust $= b$.

Writing this equation in the form,

$$b = w \tan^2 \left(45^\circ - \frac{\varphi}{2}\right) \left[x - \frac{2\,c}{w} \tan \left(45^\circ + \frac{\varphi}{2}\right)\right]$$

and putting $x' = \frac{2\,c}{w} \tan \left(45^\circ + \frac{\varphi}{2}\right)$, we have

$$b = w \tan^2 \left(45^\circ - \frac{\varphi}{2}\right) [x - x'].$$

$b = o$ when $x = x'$, which agrees with the result of Art. 92.
The equation is not valid for $x < x'$, for an unlimited mass of earth.

The planes of rupture IS and IT have their true directions, since IJ is now horizontal. Therefore the angle with the vertical IN, Fig. 62, made by either plane $= NIS = \frac{1}{2}\, IPS = \frac{1}{2}\,(90° - \varphi)$; so that *a plane of rupture bisects the angle between the vertical and the natural slope.* Since this result is true for any $x < x'$, it follows that the plane of rupture maintains the same inclination DAI, Fig. 63, with the vertical, from the level ID, where $x = x'$, down.

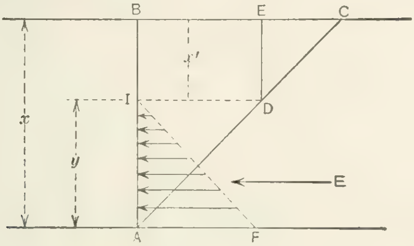

FIG. 63

In Fig. 63, where BC is the level free surface, on putting $y = x - x'$,

$$b = w\, y\, tan^2\left(45° - \frac{\varphi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is the formula for the horizontal unit thrust of non-coherent earth, with ID for a free surface.

On laying off the value of b at $A = FA$ and drawing a straight line from I to F, it is seen that the total horizontal thrust E on the vertical plane AB is represented by the area of the triangle

$$IAF = \frac{1}{2}\, yb$$

$$\therefore\ E = \frac{1}{2}\, wy^2\, tan^2\left(45° - \frac{\varphi}{2}\right) \quad . \quad . \quad . \quad (2)$$

Also, since it is a uniformly varying stress, the resultant acts at $\frac{1}{3}y = \frac{1}{3}AI$ above A. To compute the numerical value of E for a given x, we first compute,

$$x' = \frac{2\,c}{w}\tan\left(45^\circ + \frac{\varphi}{2}\right),$$

then $y = x - x'$, and finally E for the given values of c, w and φ.

For an unlimited mass of earth, subjected to no force but its own weight, it will now be shown that there is no horizontal stress on a vertical plane, in the layer $IDCB$ of Fig. 63 of depth x'.

In Fig. 64, let DE be the free surface and at the depth $x = DB$, conceive a wedge of rupture ABC, the angle BAC being $45^\circ - \frac{\varphi}{2}$, as proved above. If we conceive the wedge to be without weight, the pressures on the three faces will correspond to those at the depth x.

Let $AB = 1$, $BC = d$ and $AC = l$; also let n = unit normal reaction of earth on plane AC; thus the total normal reaction is nl, the frictional resistance is $nlf = nl \tan\varphi$ and the cohesion cl, the last two forces acting up along AC, opposite to the impending motion. The wedge is thus in equilibrium under the forces, $b \times 1$, wxd, nl and $cl + nlf$. If we lay off $N_1OL_1 = \varphi$, then if $nl = ON_1$, $L_1N_1 = nlf$ or if $nl = ON_2$, then $L_2N_2 = nlf$, etc. Prolong N_1L_1 to Q_1, making $Q_1L_1 = cl$ and draw Q_1Q_3 parallel to OL_1.

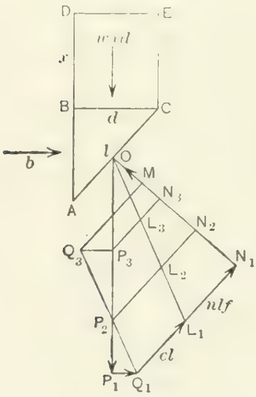

FIG. 64

Now if $wxd = OP_1$, then the horizontal P_1Q_1 represents b, since we have a closed polygon $OP_1Q_1N_1O$ whose sides represent the forces wxd, b, $cl + nlf$ and nl, in order. Here $b = P_1Q_1$, acting to the right, is compressive. When $wxd = OP_2$, the force polygon is, OP_2N_2O and $b = 0$. This corresponds to the depth x'. For a less depth, so that $wxd = OP_3$, the force polygon is OP_3N_3O, and since now $N_3O = nl$, it will be observed that $P_3N_3 < cl + nlf = cl +$

L_3N_3. Hence when $x < x'$, the full friction and cohesion is not exerted on the plane AC and its resultant reaction P_3O (the resultant of P_3N_3 and N_3O) is vertical and exactly balances the weight of the prism BE of earth resting on BC.

There is thus no necessity for tensile forces and none are exerted in the layer $IDCB$ of Fig. 63. There is then no stress on the vertical plane DE, Fig. 63, since $b = o$ for any point of it, and each vertical prism resting on DC is sustained by a vertical reaction of the plane DC; in fact, the part $IDEB$ can be removed and the prism DEC will be entirely sustained by the vertical reaction of DC. It has been seen too that only a portion of the full cohesion and friction possible is required for equilibrium, and this portion becomes proportionately less as x (or OP_3, Fig. 64) diminishes.

The case is different if the earth gives way to the left of the vertical plane AB, Fig. 63, as from the failure of a retaining wall or from digging a vertical trench at a greater depth than x'. In this case, if the earth is capable of exerting tension, the polygon of forces in Fig. 64, when $wxd = OP_3$, will be, $OP_3\,Q_3\,MO$, if the full cohesion and friction Q_3M on AC is exerted. Since P_3Q_3 is directed to the left, $b = P_3Q_3$ is tensile. If the reaction b is not horizontal, but directed upward, as is probable, the polygon will be correspondingly changed.

Recurring now to the case of the unlimited mass of earth, where there is no stress on ED, Fig. 63, the prism EDC being self supporting, it is seen that the only resistance to sliding of the prism $ADEB$, down the plane AD, is the total cohesion and friction exerted on AD alone. Stating the conditions of equilibrium for the prism $ADEB$, subjected to its own weight, the total reaction on AD, and the thrust on AB, acting from left to right, it can be proved that the plane of rupture AD makes the angle $IAD = \left(45° - \frac{\varphi}{2}\right)$ with the vertical and that the thrust is exactly the value of E given above. The proof, being long, is omitted.

If, however, the earth gives sufficiently to the left of AB, then (1) it may be supposed that tensile forces will be exerted along *ED* that will drag the whole or a part of the prism *DEC*, down the plane *DC* and either the whole or a part of the cohesion and friction that can be exerted along *DC* will be called into play.

If the full amount of friction and cohesion possible is exerted along *DC*, then the wedge of rupture is no longer *ADEB*, but *ACB*. Stating the conditions of equilibrium for *ACB*, it can be proved that the equilibrating horizontal thrust is,

$$E' = \frac{w}{2} x^2 \tan^2 \left(45^\circ - \frac{\varphi}{2}\right) - 2\, cx \tan \left(45^\circ - \frac{\varphi}{2}\right)$$

The state of stress throughout the mass is now complex, involving tensile, compressive, and shearing forces.

When $E' = o$, we derive,

$$x = \frac{4\,c}{w} \tan \left(45^\circ + \frac{\varphi}{2}\right)$$

which is double the height x' found above. This formula will be proved independently in the next article.

(2) It may be supposed that the tensile strength of the earth along *DE*, Fig. 63, is not sufficient to drag *EDC* down, but that the impending motion of *ADEB* is resisted not only by the friction and cohesion along *AD*, but also by the cohesive resistance acting upward along *DE*.

(3) After heavy rains, the earth on drying out, causing contraction near the surface, may develop vertical cracks, so that not even cohesive resistances can be assured along *DE*.

It is, perhaps, a common fact of observation, when slides occur in trenches or open cuts, in ordinary clayey earth, that the surface of rupture is nearly vertical for several feet down from the top, and below this it is much less inclined to the horizontal. The lower part of the break is often curved and concave upwards.*

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* In the discussion of Mr. J. C. Meem's paper on "The Bracing of Trenches and Tunnels," *Trans. Am. Soc. C.E.*, vol. LX, pp. 1—100, a great many facts of observation, vital to the constructor, are recorded. In this connection, see especially Mr. E. G. Haines's remarks on surfaces of rupture.

Such facts lend color to the belief that either assumptions (2) or (3) are more probable than (1). To be on the side of safety, (3) is suggested, so that it is well not to count on a greater height than x' for the unbraced trench, particularly for a period of years.

An important case, often met with in practice, is that of the retaining wall, backed by earth level with the top of the wall, the earth being loaded uniformly. Conceive the surcharge to consist of the same weight of earth whose specific weight is that of the filling, the height of such a surcharge being x_0. Let the inner face of the wall pass through A, Fig. 63, and lie to the left of the vertical through A and suppose it required to find the earth thrust on the vertical plane from A to the surface of the filling, the thrust acting horizontally.

Let BC represent the top of the surcharge, so that the horizontal surface of the filling is at a distance x_0 below BC. Then if $x_0 \leqq x'$, or if the surface of the filling is at ID or above it, then the thrust E required is found from (2) above, since AD, where the resistance to sliding is exerted, lies wholly in the filling. This is not true when $x_0 \geqq x'$, for then a part of AD lies in the surcharge which offers no resistance to sliding. In this case, compute by (1), the unit pressure at depth x_0 or at the surface of the filling; also compute by (1) the unit pressure for $x = AB$ or from A to the surface of the surcharge. The pressure increases uniformly from $x = x_0$ to $x = AB$ and the resultant thrust and its point of application can be found as in the corresponding problem relating to Fig. 52, Art. 88.

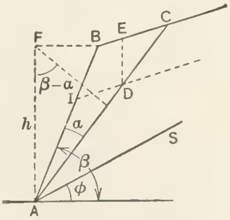

Fig. 65

97. Trench with Sloping Face. In Fig. 65, AB represents the face of the trench of vertical height h and inclined at an angle β to the horizontal. In the usual theory, any trial plane of rupture AC, inclined at the angle α to AB, is supposed to exert frictional and cohesive resistances throughout its whole extent to impending motion downward of the wedge ABC.

The weight W of $ABC = \frac{w}{2} AB.AC \sin\alpha$

$$= \frac{w}{2} \frac{h}{\sin\beta}.AC \sin\alpha.$$

The components of W, parallel and perpendicular to AC are, $W \sin(\beta - \alpha)$, $W \cos(\beta - \alpha)$, respectively and the frictional resistance along AC will be, $W \cos(\beta - \alpha) \tan\varphi$.

The cohesive resistance $= c' \times AC$, if we call c' the cohesion per unit area on the plane AC. For the true plane of rupture, the cohesion is c, the maximum, or coefficient of cohesion. For any other plane as AC, $c' < c$ and c' varies with α.

Balancing components parallel to AC, for equilibrium,

$$W[\sin(\beta - \alpha) - \cos(\beta - \alpha)\tan\varphi] = c' \times AC.$$

The [ ] reduces to $\frac{\sin(\beta - \alpha - \varphi)}{\cos\varphi}$.

Substituting, replacing W by its value and dividing by AC,

$$\frac{w}{2} \frac{h \sin\alpha}{\sin\beta} \frac{\sin(\beta - \alpha - \varphi)}{\cos\varphi} = c'.$$

This is a maximum when $\frac{dc'}{d\alpha} = 0$.

$$\therefore \cos\alpha \sin(\beta - \alpha - \varphi) - \sin\alpha \cos(\beta - \alpha - \varphi) = 0$$

$$\therefore \tan\alpha = \tan(\beta - \alpha - \varphi) \therefore \alpha = \beta - \alpha - \varphi.$$

$$\therefore \alpha = \frac{\beta - \varphi}{2}$$

or AC bisects the angle $(\beta - \varphi) = BAS$.

On substituting this value of α in the value of c', which now becomes c, we find

$$wh \sin^2 \frac{\beta - \varphi}{2} = 2c \sin\beta \cos\varphi.$$

To criticize this well-known formula for computing the maximum height h for a stable slope, let the vertical $DE = \frac{2c}{w} \tan\left(45° + \frac{\varphi}{2}\right)$; then before excavating the trench, there is no stress on DE, Art. 92. After the earth is removed to the left of AB, the mass $ADEB$ would slide down AD, unless the earth along DE could exert sufficient tension to drag down the whole of the wedge EDC, which is improbable, since observation shows that the upper part of a slide is nearly vertical. Hence the surface of rupture is not ADC, but say ADE or a curved surface lying perhaps near it. Hence the true height for an unsupported trench or open cut is less than is given by the above formula.

To deduce a formula, where the trial prism of rupture is of the type $ADEB$, let i = inclination of BC to the horizontal and denote DE by x'; also draw from D a parallel to BC intersecting AB at I. Replace the area $IDEB$ by the parallelogram (sides $\|$ ID and ED) whose area is $ID \times DE \cos i$, which is on the side of safety. The weight of the prism $ADEB$ is thus,

$$w \, . \, ID \left[\tfrac{1}{2} AI \sin (\beta - i) + x' \cos i\right] =$$

$$w \, . \, AD \frac{\sin \alpha}{\sin (\beta - i)} \left[\tfrac{1}{2} AI \sin (\beta - i) + x' \cos i\right].$$

This expression replaces W in the first equation of equilibrium above and $c'.AD$, replaces $c'.AC$. After dividing the resulting equation by AD, we have,

$$\frac{w \, . \sin \alpha \, \sin (\beta - \alpha - \varphi)}{\sin (\beta - i) \cos \phi} \left[\tfrac{1}{2} AI \sin (\beta - i) + x' \cos i\right] = c'.$$

On differentiating with respect to α, we find as before, for a maximum c', $\alpha = \tfrac{1}{2} (\beta - \varphi)$; whence substituting and solving for AI,

$$AI = \frac{2\,c}{w} \frac{\cos \varphi}{\sin^2 \dfrac{\beta - \varphi}{2}} - 2\,x' \frac{\cos i}{\sin (\beta - i)}$$

Also, since $h = AI \sin \beta + IB \sin \beta =$

$$AI \sin \beta + x' \frac{\sin \beta \cos i}{\sin (\beta - i)},$$

we reach the final formula,

$$h = \frac{2\,c}{w} \frac{\cos \varphi \sin \beta}{\sin^2 \dfrac{\beta - \varphi}{2}} - x' \frac{\sin \beta \cos i}{\sin (\beta - i)},$$

where, $x' = \dfrac{2\,c}{w} \tan \left(45^\circ + \dfrac{\varphi}{2}\right)$.

It will be observed that this value of h only differs from that first found in the subtractive term. As an application let $i = 0$ $c = 300$ lb./ft.^{2}, $w = 112$ lb./ft.^{3}, $\varphi = 8^\circ$, $\beta = 33^\circ\,41'$, whence by the formula $h = 53.6$ ft. This only differs $x' = 6.16$ ft. from the result of the first formula.

When c is taken as 400 lb./ft.^{2}, the other quantities remaining the same, we find $h = 71.4$ ft. In the reservoir embankment of Charmes,* which failed, the experimental determinations were $\varphi = 8^\circ$, $\beta = 33^\circ\,41'$, $c = 400$ lb./ft.^{2} (395 to 448), $w = 112$ lb./ft.^{3}, $h = 56$ ft. or less than 71.4 as found by the formula. Apparently the formula still gives an excess height, particularly so since the lower 20 ft. of the embankment was covered with water, whose hydrostatic pressure added to the stability. The surfaces of rupture were somewhat of the shape shown in Fig. 66, by the dotted line. The ruptures were attributed to the lowering of the water level.

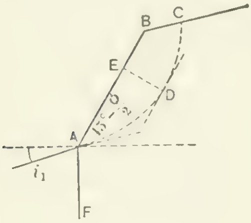

FIG. 66

If in the last formula, we make $\beta = 90^\circ$, we have for the maximum height, h_o at which a vertical trench will stand unsupported, when $i = 0$,

$$h_o = \frac{2\,c}{w}\,\frac{\sin\,(90^\circ - \varphi)}{\sin^2\left(45^\circ - \frac{\varphi}{2}\right)} - x' =$$

$$\frac{4\,c}{w}\,\frac{\sin\left(45^\circ - \frac{\varphi}{2}\right)\cos\left(45^\circ - \frac{\varphi}{2}\right)}{\sin^2\left(45^\circ - \frac{\varphi}{2}\right)} - x'$$

$$\therefore\ h_o = \frac{4\,c}{w}\tan\left(45^\circ + \frac{\varphi}{2}\right) - x' = \frac{2\,c}{w}\tan\left(45^\circ + \frac{\varphi}{2}\right).$$

As before remarked, this ignores any possible tension in the earth along DE, Fig. 65. If sufficient tension exists to drag down the wedge *EDC*, then the last value of h_o given will be doubled, since the term x' is then omitted.

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* Résal, *Poussée des Terres*, II., pp. 327–340.

A further restriction must be placed upon the angle i that the surface *BC* makes with the horizontal, namely, that generally, $i \leqq \varphi$; for if $i > \varphi$, and if a line be drawn parallel to *BC* at the critical depth x_o, Art. 91, and this line intersects *AB*, there will be sliding along the corresponding surface at the depth x_o below *BC*. Generally (or when β is not near 90°) $x_o < h$, so that i should be $\leqq \varphi$ for stability.

Considering the face *AB*, Fig. 66, making the angle β with the horizontal, the critical depth normal to *AB*, is Art. 91,

$$ED = x_o \cos \beta = \frac{c}{w} \frac{\cos \varphi}{\sin (\beta - \varphi)}$$

This is strictly true only when *AB* is unlimited and is only approximately true for usual heights of trench. Sliding is then imminent in the vicinity of point *D*, Fig. 66, the surface of rupture passing through *A* and having about the shape *ADC*, being curved and concave upward. Résal,[*] who suggests this curve *ADC*, further states that it should make an angle of $45° - \frac{\varphi}{2}$ with *AB* at *A*, since the plane of rupture, for passive resistance of the earth at *A*, makes this angle with the surface regarded as *AB*. Thus in Fig. 61, the curve $A'T'$ makes the angle $\left(45° - \frac{\varphi}{2}\right)$ with $A'B$ at A'.[†]

The bottom of the trench to the left of *A* can make an angle with the horizontal i_1, provided $i_1 \leqq \varphi$, for in that case the passive resistance of the earth to the left of a vertical plane AF is sufficient to overcome the active thrust, on this plane, of the earth to the right of FAB; but this is not true when $i_1 > \varphi$, since at the critical depth $x_0 = AF$ (putting i_1 for i in formula) the equilibrium is unstable and for a greater depth there is no equilibrium, so that heaving of the earth in the bottom of the trench would occur.

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[*] *Poussée des Terres,* II., p. 95.

[†] In applying the formula for *ED* to the Charmes embankment, which was composed of heterogeneous materials and constructed after the most approved manner, it was thought at *D*, the earth being firmer and less saturated than near the surface, that $\varphi = 9° 40'$ and possibly $c = 800$ lb. ft.^{2}, would represent more nearly the constants. On substituting in the formula, we find $ED = 17.3$ ft. In the actual slide *ED*, as measured, was 24.6 ft., the surface of the slide taking the general shape of the dotted curve in Fig. 66 and emerging at the upper surface nearly at right angles to it.

It will be observed from the formula for ED, Fig. 66, that when $\beta = 90°$, $ED = c/w$.

It may readily be surmised from what precedes that the theory of stable slopes is not on a very satisfactory basis, but its indications may be of service to the engineer when coupled with his practical judgment.

98. Earth Surface Horizontal. Passive Thrust. Referring to Art. 96 and Fig. 62, it is seen that when $r = OI = b = wx$, then the passive horizontal thrust, $r' = OJ = a$; hence on solving an equation in Art. 96, for a, we derive,

$$a = b\ tan^2\left(45° + \frac{\varphi}{2}\right) + 2\ c\ tan\left(45° + \frac{\varphi}{2}\right)$$

$$\therefore \text{ putting } x'' = \frac{2\ c}{w} cot\left(45° + \frac{\varphi}{2}\right),$$

$$a = w\ (x + x'')\ tan^2\left(45° + \frac{\varphi}{2}\right).^*$$

99. Foundations. In Fig. 67 is shown a wall backed by earth. Let W = vertical component of the resultant on the earth foundation BC in pounds and A = area base BC in square feet, both for 1 foot length of wall. If the resultant cuts the base at its center, the vertical unit pressures p and p' at the toe and heel of the wall, each equal W/A; otherwise they can be computed from a formula given in Art. 15, so that it will be assumed that both are known for any position of the resultant.

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* If earth is placed in front of a stable wall, only its active thrust is exerted against the wall. If, however, the wall slides, even a very small distance, the full passive resistance is brought into play, which, for earth level at top, can be computed by aid of the above formula, which gives the unit pressure at the surface for $x = 0$ or generally, for any depth x, the pressures varying as the ordinates of a trapezoid from the surface down. The resultant horizontal pressure thus acts through the center of gravity of the trapezoid.

Let w_1 and w_2 denote the weight per cu. ft. of the earth in front of the wall and of the filling back of the wall respectively.

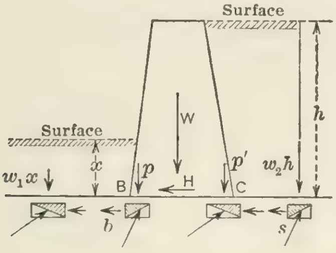

FIG. 67

The coefficients c and $f = \tan \varphi$, refer to the earth of the wedges of rupture just below BC.

The active thrust b at the toe of the wall, caused by the pressure p, tending to cause the little wedge of rupture to slide down, is, by a formula of Art. 96,

$$b = p \tan^2\left(45^\circ - \frac{\varphi}{2}\right) - 2\,c \tan\left(45^\circ - \frac{\varphi}{2}\right).$$

If x denote the depth from the surface of earth in front of the wall to B, then if b is sufficiently large to develop the full passive resistance of the earth just to the left of B, Art. 98,

$$w_1\, x \tan^2\left(45^\circ + \frac{\varphi}{2}\right) + 2\,c \tan\left(45^\circ + \frac{\varphi}{2}\right),$$

or to cause impending motion of the corresponding wedge of rupture up the plane of rupture, so that the earth is about to heave or rise, then the active thrust is equal to the passive thrust. Placing them equal and solving for p, we have,*

$$p = w_1\, x \tan^4\left(45^\circ + \frac{\varphi}{2}\right) + 2\,c \tan^3\left(45^\circ + \frac{\varphi}{2}\right) + 2\,c \tan\left(45^\circ + \frac{\varphi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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* Reducing by relation, $\tan\left(45^\circ - \frac{\phi}{2}\right) = \cot\left(45^\circ + \frac{\phi}{2}\right) = \dfrac{1}{\tan\left(45^\circ + \frac{\phi}{2}\right)}.$

The unit toe pressure must not exceed this value of p, or heaving of the earth in front of the wall will occur.

For a given toe pressure p the equation may be solved for x, the least height the earth should extend above B to prevent heaving.

At the heel of the wall C, where the height of filling is h, the active horizontal unit thrust due to the filling is,

$$w_2 h \tan^2\left(45^\circ - \frac{\varphi}{2}\right) - 2c \tan\left(45^\circ - \frac{\varphi}{2}\right).$$

This should not exceed the passive unit resistance to sliding up the plane of rupture shown in the figure just below C,

$$p' \tan^2\left(45^\circ + \frac{\varphi}{2}\right) + 2c \tan\left(45^\circ + \frac{\varphi}{2}\right),$$

or the earth will heave the wall.

Equating the active and passive thrusts and solving for p',

$$p' = w_2 h \tan^4\left(45^\circ - \frac{\varphi}{2}\right) - 2c \tan^3\left(45^\circ - \frac{\varphi}{2}\right) - 2c \tan\left(45^\circ - \frac{\varphi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To prevent heaving or tilting of the wall, p' should not be less than the value given by (2).

In deriving (1) and (2), the horizontal component H of the resultant thrust on BC has been neglected, though it undoubtedly exercises an influence on the pressures both at B and C.

If the actual pressure at the heel of a retaining wall is less than the value given by (2), the wall will tilt over somewhat; likewise if the pressure at the toe is greater than the value given by (1), the earth in front of the wall will heave and the toe may sink slightly. If the increased leaning of the wall from one or both causes is not too great, its decreased stability may be so small that the wall remains perfectly stable.

In fact, heavy weights, as machinery, road rollers, etc., are constantly being placed on the surface of the ground without mishap. However, the soil is compressed somewhat and the earth may heave a little on the sides. Similarly, the weight of a string of locomotives, on a narrow embankment, is safely transmitted to its base. Imagine a wall to replace the locomotives. The earth is likely to settle more than where earth is filled in outside the embankment, either level with its top or rising above it, as in Fig. 67. In the latter case, when x is greater than the value given by (1), for a given p, there can be no sinking at the toe, from the little wedge of rupture there sliding down.

It is true that most of the weight of the wall is sustained by the earth occupying the space of the original embankment, still it is desirable, for a stable wall, to have no sinking of the kind supposed and consequently, the formulas above may prove useful in ascertaining permissible soil pressures p and p' when the constants w_1, w_2, c and φ, have been determined by experiment.*

EXAMPLE. Referring to Fig. 67, let $w_1 = w_2 = 100$ lb./ft.^{3}, the specific weight of earth before and behind the wall. For the earth of the foundation, just below level BC, let $\varphi = 15°$, $c = 200$ lb./ft.^{2}. Also let $x = 3$ ft. $h = 20$ ft.

Therefore by (1) the pressure p, at the toe, must not exceed 2271 lb./ft.^{2}, or the earth in front of the wall will heave, and by (2) the pressure p' at the heel must not be less than 205 lb./ft.^{2} or the earth will heave the wall. When (2) gives p' negative, it indicates that the active thrust is less than the passive, so that no heaving of the wall can occur even when there is no pressure of the masonry at the heel.

100. Screw Pile. The greatest unit pressure p, acting vertically, that can be sustained by the earth at the base of a screw pile, without heaving of the earth or settlement of the pile, is given by (1) of Art. 99, x being the depth from the surface down to the screw.

Thus for a moist, clayey earth, suppose $\varphi = 15°$, $w_1 = 100$ lb./ft.^{3}, $c = 400$ lb./ft.^{2}, then from (1), Art. 99, we find,

$$p = 288\,x + 2768 \text{ lb./ft.}^2$$

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* See paper by Mr. A. L. Bell on "The Lateral Pressure and Resistance of Clay and the Supporting Power of Clay Foundations," *Min. Proc. Inst. C. E.*, vol. CXCIX, 1914–1915, Part *i*, where similar formulas to the above were first given and applied in the manner indicated.

If A = area horizontal projection of the screw in square feet, the load the pile can just carry is $p\,A$ pounds. The same formula applies in finding the unit pressure that can be sustained by the earth at the foot of a cylindrical pile. The friction exerted on the sides of the pile from the earth pressure cannot be computed with any accuracy, possibly from a polishing of the earth next the pile and a consequent large decrease there in the coefficients c and f as the pile is driven down or sinks. The sustaining power due to the friction of the earth on the vertical sides of the pile is usually estimated empirically.

101. Thrust Against a Retaining Wall in the Case of Coherent Earth. If we proceed in a manner analogous to the usual Rankine method, to find the thrust against the battered wall of Fig. 67, the thrust which acts on a vertical plane through the heel C is first ascertained and this thrust is then combined with the weight of earth lying between the vertical plane and the wall, to get the thrust on the latter. When the surface is horizontal, the horizontal thrust on the vertical plane is given by the value of E, Art. 96, and it acts $1/3\,(h - x')$ above C, since it is uniformly increasing from the depth x' (where it is zero) to the heel.

If we conceive a plane drawn parallel to the surface and at a depth $x' = \frac{2\,c}{w}\,tan\left(45^\circ + \frac{\varphi}{2}\right)$, below it, and regard this plane as the free surface of non-coherent earth of the same specific weight and angle of friction as the given earth and compute the thrust on the vertical plane for such earth, the resulting formula is found to be identical with the value for E for the coherent earth just quoted.

The same conclusion holds approximately when $i < \varphi$, if the surface is inclined at the angle i to the horizontal. Thus for the example of Art. 93, footnote, where $i = 15^\circ$, $\varphi = 30^\circ$, $c = w = 100$, $x' = 3.46$ ft., at the vertical depth $x = 517.50$ ft., the unit thrust as given by the circular diagram, is found to be, $r' = 19{,}200$ lb./ft.^{2}. By Rankine's formula, Art. 48, for $x = 517.50 - 3.46 = 514.04$, the thrust is the same. At the depth 31.05 ft. by circular diagram, $r' = 1020$, but for $x' = 31.05 - 3.46 = 27.59$ ft., the Rankine formula gives $r' = 1029$ lb./ft.^{2}

As we have seen, when i is not zero, the unit thrust is not exactly uniformly increasing, but it can be assumed as such with all desirable accuracy, when $i < \varphi$; so that the resultant thrust can be assumed to act $\frac{1}{3}(x - x')$ above C, Fig. 67, where x is the vertical distance from C to the inclined surface of the earth.

By reference to Fig. 64, referring to the case where $i = o$, $\angle BAC$ constant for any $x > x'$, it is seen that P_1Q_1 varies lineally with P_1P_2 or the conjugate thrust varies uniformly as $(x - x')$. Similarly it might be inferred when $i < \varphi$, since the inclination of the surface of rupture, Fig. 60, varies but slightly with x, that r', the conjugate thrust varies nearly uniformly with $(x - x')$, being zero at $x = x'$. This is easily seen from Fig. 64, regarding $\angle BAC$ as constant, on drawing P_1Q_1 parallel to the inclined surface to represent r'.

However, when i is very near φ and especially when $i = \varphi$, r' is not found to vary lineally (see Ex., Art. 89), the surface of rupture being considerably curved. The resultant thrust acts a little below the third point, so that it is safe to regard it as acting at the third point, but the unit thrust at C, Fig. 67, must be found from the circular diagram (see remark in Art. 89) and not by the approximate method outlined above. When $i > \varphi$, the curve AS, Fig. 61, differs still more from a straight line and the linear law does not obtain. The total thrust, when $x \leqq x_o$, may be computed as explained in Art. 94 and its position found, either by taking moments, or assuming, for abundant security, that it acts $\frac{1}{3}(x - x')$ above the base.

It will be recalled in connection with Fig. 61, that for a depth greater than

$$x_o = BC = \frac{c}{w \cos i} \, \frac{\cos \varphi}{\sin (i - \varphi)}$$

slipping would occur along AS if the earth was free to move. However, if a wall, of height greater than x_o, retains the earth, the reaction of the wall introduces forces not contemplated in the theory of the indefinite mass of earth subjected only to its own weight and the previous solution does not strictly apply.

It seems reasonable to suppose that heaving of the earth would occur; but ignoring that, an approximate estimate of the thrust can be made by supposing the conjugate active thrust r' to increase uniformly from $r' = 0$ at $x = x'$ to the value given by the circular diagram or formula at $x = x_0$ and that this rate of increase continues to the bottom of the wall. Thus for the example of Art. 94, footnote, where $i = 25^\circ$, $\varphi = 20^\circ$, $c = 200$ lb./ft.^{2}, $w = 100$ lb./ft.^{3}, $x' = 5.71$ ft., $x_0 = 23.80$ ft., $r' = 1990$ lb./ft.^{2} at $x = x_0$, r' can be assumed to increase uniformly from o at $x = x'$ to 1990 at $x = x_0$ and to reach the value $\dfrac{45.71 - 5.71}{23.80 - 5.71} \times 1990 = 4400$ lb./ft.^{2} at $x = 45.71$ ft. Compare with the result of the last example in Art. 102.

The indefinite slope does not occur in practice for filling deposited after the wall is built, since embankments are rarely over 150 ft. in height. In this case, at first, the surface slopes perhaps nearly at the angle of repose for non-coherent earth. In course of time, the cohesion of the earth may be very much increased and the earth thrust diminished; but suppose the earth at some time saturated with water which may still further increase the cohesion, though it may diminish φ, the angle of friction; then the thrust against the wall may be greater than in the first instance. Thus in the Charmes reservoir dam, after more or less saturation of the earth with water, φ was found experimentally to be only 8°, $c = 400$ lb./ft.^{2} Presumably, $33^\circ\,41'$ was the value of φ for the dry pulverulent earth when first deposited.

Comparing the horizontal thrusts on a vertical plane, for earth surface horizontal, for case (1), where $c = o$, $\varphi = 33^\circ\,41'$, $w = 100$ (say), and case (2), where $c = 400$, $\varphi = 8^\circ$, $w = 112$ lb./ft.^{3}, we find the thrusts the same at the depth $x = 19.6$ ft.; but at the depth 46.3 ft., the resultant thrust for the earth of case (2) is double that for case (1). However, for case (2), it acts 12.7 ft. above the base, whereas for case (1) it acts 15.4 ft. above the base.

Such facts offer perhaps an explanation of the failure of retaining walls that are not drained and of bridge abutments, where the filling is saturated at high water. Such filling, for a little distance back of the abutment, should preferably be of gravel, whose angle of friction is not much affected by water. In the case of a face wall at the foot of a long mountain slope, thorough drainage is essential.

It may be further remarked that in any case when $i > \varphi$, the dangerous height, Art. 97, should not be exceeded, otherwise the earth above the foot wall may rupture and the approximate prism of rupture slide over the wall.

A novel device for retaining the earth of a railway embankment, originated by Mr. Gustav Lindenthal, is described in *Engineering News* of May 6, 1915, p. 886. The greatest height of the embankment is 65 ft., width 57 ft. 6 in., and it is retained by two thin vertical concrete walls, 57.5 ft. apart, which are connected every 10 ft., vertically and horizontally, by tie rods 2¼ in. diameter, which are anchored into the concrete by longitudinal and vertical 8-in. 16¼-lb. channels. The earth was rammed as it was filled in and was thoroughly drained with vertical wells and horizontal drains. The earth thrust was estimated somewhat arbitrarily. A more correct way, for this very coherent earth, would be to put specimens of the earth that had been rammed between the plaques and determine c and φ, as explained in Art. 3, and then use the method given at the end of Art. 96 for a surcharged filling, in estimating the earth thrusts at different depths.

Remark. In using the above methods of estimating the stability of a retaining wall, it is understood that the coefficients f and c are to be found by experiments on the particular filling used. Then the stability of the wall can be *tested* in the usual manner. As to whether the wall should be *designed* for the coherent earth depends entirely upon the degree of permanence of the coefficients f and c.

It is known that clay is a treacherous material, which has not permanent characteristics, so that its use as a filling is not to be recommended; but it is possible that earth containing sand and clay with over 50% of sand, well rammed as it is filled in and thoroughly drained by both vertical wells and horizontal drains, may show, under the stress of all kinds of weather, so little variation in the coefficients that they may be regarded as permanent, and thus the wall can safely be designed for the particular coherent earth.

In the design of the temporary bracing of trenches, the theory for coherent earth may be used, but experience shows that in time—especially after heavy rainfalls—the values of f and c are materially altered, so that bracing so designed cannot be regarded as a permanent feature.

For earth—even clay—not exposed to the atmosphere, as in the foundations of walls, the permanency of the coefficients can be more confidently relied on.

102. Graphical Method of Estimating the Total Thrust of Coherent Earth Against a Retaining Wall. The following method of estimating the total thrust against a wall is more general than the preceding, in that it includes the influence of the friction of the earth on the wall; besides it is applicable to walls leaning toward the earth and to any kind of a surcharged wall. Although the construction below is given for a particular case, the method is general and is easily applied to any case of a battered, leaning, or surcharged wall.

In Fig. 68, let AB be the inner face of the wall, retaining the earth to its right, whose surface is represented by the indefinite line Bb_4. The line AH is horizontal.

Let φ = angle of friction, c = coefficient of cohesion in lb./ft.^{2} and w = weight of a cubic foot of earth in pounds.

(1) Compute, $x' = \frac{2c}{w} \tan\left(45° + \frac{\varphi}{2}\right)$ and lay it off vertically downward from the free surface. Through the lower extremity of this line draw a parallel $F4$ to the free surface.

(2) Lay off a convenient length $F1$ and equal lengths $12 = 23 = 34$, along $F4$, erect verticals $1b_1$, $1\ b_2$, . . ., and compute

the areas $A1\,b_1\,B$, $A2\,b_2\,B$, . . . Thus, if p is the length of the perpendicular from A upon $F4$, area $A1\,b_1\,B = \frac{1}{2}\,p \,.\, F1 + \frac{1}{2}\,(F1 + Bb_1)\,q$, where q = perpendicular distance between $F4$ and Bb_4. Also, area $A2\,b_2\,b_1\,1\,A = \overline{12}\,(\frac{1}{2}\,p + q)$ = area $A3\,b_3\,b_2\,2\,A$ = area $A4\,b_4\,b_3\,3\,A$. By continued addition, the

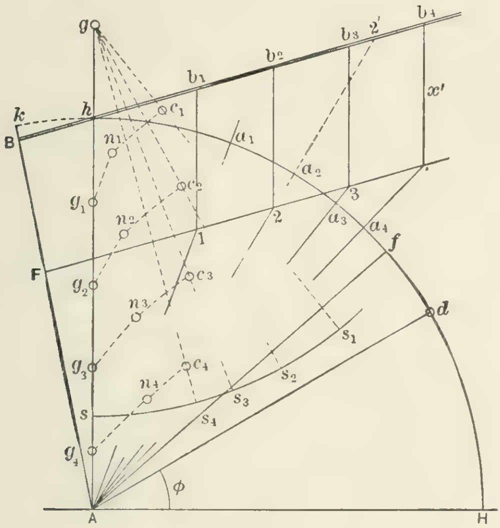

FIG. 68

areas required are ascertained, and on multiplying by w the weights of the trial prisms of rupture,

$$A1\,b_1\,B,\ A2\,b_2\,B,\ A3\,b_3\,B,\ A4\,b_4\,B,$$

are found.

On a convenient vertical, as Ag, lay these weights off to scale from g : gg_1, gg_2, gg_3, gg_4, representing the successive weights.

(3) Measure the lengths $A1$, $A2$, $A3$, $A4$, to the scale of distance in feet and multiply each length by c. From g_1, draw $g_1\,n_1 \parallel A1$ and $= A1 \,.\, c$, to the scale of loads, to represent the cohesion, acting upward along $A1$. Similarly, draw $g_2\,n_2 \parallel A2$ and $= c \,.\, A2$, $g_3\,n_3 \parallel A3$ and $= c \,.\, A3$, $g_4\,n_4 \parallel A4$ and $= c \,.\, A4$, to represent the cohesion along $A2$, $A3$ and $A4$, in turn.

(4) With A as a center and a convenient radius, as Ah, describe the arc $kh\ H$. With g as a center and the same radius,

describe the arc ss_1. Lay off $HAd = \varphi$ and mark the intersections, $d_1, d_2, \ldots$, of $A1, A2, \ldots$ (produced where necessary) with the arc $kd\,H$. Then, with dividers, lay off chord $ss_1 =$ chord da_1, $ss_2 = da_2, \ldots$, whence the lines $gs_1, gs_2, \ldots$ make angles φ with the normals to planes $A1$, $A2$, $A3$, $A4$, Art. 19.

(5) If we assume the angle of friction of earth on wall φ' to equal φ, lay off chord $df =$ chord hk, whence fA gives the direction of the earth thrust on the wall, making the angle φ with the normal to AB. Next, draw through $n_1, n_2, \ldots$ lines parallel to Af to intersections $c_1, c_2, \ldots$, with $gs_1, gs_2, \ldots$, respectively. Pick out, with dividers, the longest of the lines nc, which is found to be $n_2 c_2$ in this figure. This line, to the scale of loads, represents the earth thrust approximately, in pounds, and $A2$ is the corresponding plane of rupture. For greater accuracy, other trial planes of rupture lying between $A1$ and $A3$ should be investigated as before, to find the maximum nc, otherwise the thrust may be slightly underestimated.

Proof: It will be observed that if we regard $A2$ as the plane of rupture, the prism of rupture $A2\,b_2\,B$ is acted on by its weight, represented by gg_2, the cohesion acting up along $A2$, represented by $g_2 n_2$, the reaction of the wall acting parallel to Af and the resultant of the normal reaction N of the plane $A2$ and the friction $N \tan \varphi$ acting up along $A2$. The resultant of N and $N \tan \varphi$ makes an angle φ with the normal to $A2$. It thus has the direction $s_2 g$ or $c_2 g$.

The sides of the closed polygon $gg_2\,n_2 c_2 g$, thus represent the four forces in equilibrium acting on the prism $A2\,b_2\,B$: gg_2 to scale equaling its weight, $g_2\,n_2$, the cohesion acting up along $A2$, $n_2 c_2$, the wall reaction and $c_2 g$ the resultant of N and $N \tan \varphi$.

Similarly, the sides of the polygons,

$$gg_1\,n_1\,c_1\,g,\ gg_3\,n_3\,c_3 g,\ \ldots,$$

represent the forces acting on the prisms

$$A1\,b_1\,B,\ A3\,b_3\,B,\ \ldots,$$

each treated in turn as the true prism of rupture.

As there can be only one true prism of rupture, it will now be proved that it is the trial prism corresponding to the greatest

nc, $n_2 c_2$ in this instance. This is true, because, since n_2 is fixed, for any less thrust than $n_2 c_2$, the point c_2 falls to the left of its first position and the new $c_2 g$, representing the part of the reaction of plane $A2$, due to N and N *tan* φ only, will thus make a greater angle than φ with the normal to $A2$, which is inconsistent with the laws of stability of earth. Since $n_2 c_2$ is thus proved to be the true wall reaction (equal and opposed to the earth thrust), the other trial thrusts, $n_1 c_1$, $n_3 c_3$, $n_4 c_4$, must now be lengthened to equal $n_2 c_2$. The new points c_1, c_3, c_4, will thus lie to the right of the old points, hence the new gc_1, gc_3, gc_4, will all make angles less than φ with the normals to the planes $A1$, $A3$, $A4$, so that stability is everywhere assured and motion downward is only impending along the plane $A2$ of the mass $A2\, b_2\, B$.

It will be recalled that in an indefinite mass of earth, no stress exists along such planes as $1b_1$, $2b_2$, etc., which is in agreement with the above construction. Similar conditions pertain to an incompressible earth and with a rigid wall upon an incompressible foundation; but for actual earths and walls on compressible foundations, there will be perhaps an appreciable moving over of the top of the wall. In this case the earth, in a tendency to descend, will exert tension along $2b_2$, etc., if it is capable of exerting tension.

Suppose a plane, as $A2$, is extended to meet the free surface, say at $2'$, then if the earth, in its impending descent, can exert sufficient tension along $2b_2$ to drag down the wedge $2b_2\, 2'$, the full cohesion and tension will be exerted along the whole length of the plane $A2'$ and similarly for other trial planes of rupture. On dealing with the forces acting on the *wedges* of rupture of the type $A2'\, B$, cohesion acting along the entire length $A2'$, the construction otherwise proceeds as before and it necessarily leads to a less earth thrust.

Although this minimum thrust may sometimes be exerted, it seems unwise to depend on it, for after heavy rains and the subsequent drying out of the earth, contraction occurs and vertical cracks often appear, particularly in clayey earth; consequently tension along the vertical planes cannot be safely allowed. The construction as given in Fig. 68 is therefore recommended. By this construction, for level-topped earth of indefinite extent, the thrust on a vertical plane Ah is horizontal and exactly equal to that given by the formula for E of Art. 96.

When $\overline{Ah} = 2x' = 2 . \overline{b_2 2}$, the thrust is appreciable, but if the second method above, corresponding to wedges of rupture of the type $A2'h$ is assumed, it will be found that the thrust on Ah is zero. In other words, when sufficient tension is exerted along vertical planes, as $1b_1$, $2b_2$, . . ., a vertical bank will be self-supporting when of height $2x'$; whereas if all possible tension is ignored or in case of vertical cracks, the bank is only self-supporting for the height x' or one-half the former height. This is in exact agreement with the results of Art. 97.

EXAMPLE. In Fig. 68, let Ah = 10 ft. and suppose the inner face of the wall AB to be battered 2 ft. in 10 ft.; take $i = 15°$, $\varphi = 30°$, c = 100 lb./ft.^{2}, w = 100 lb. ft.^{3} and find x' = 3.46 ft. and E = thrust on AB, making the angle φ with the normal to AB, 1200 lbs.

When AB, Fig. 68, is vertical and the thrust is taken as acting parallel to the surface, the results by the construction can be compared with those given by the circular diagram. Since the construction of Fig. 68 assumes a plane surface of rupture whereas the true surface of rupture is curved except when $i = o$, it is to be expected that the thrusts as found by the two methods will agree for $i = o$, and will differ more and more as the surface of rupture departs more and more from a plane. As observed above, the results for $i = o$ are identical. As i increases, the thrusts by the two methods remain practically the same until i is very nearly equal to φ. When $i < \varphi$, the true surface of rupture, AS, Fig. 60, is nearly plane.

Thus for $i = 15°$, $\varphi = 30°$, $c = w = 100$, the construction of Fig. 68, gives the thrust on Ah = 43,200 lbs., when Ah = 51.75 ft. The circular diagram gives the unit thrust at this depth = 1790 lb. ft.^{2}, it being zero at x = 3.46 ft. Hence the total thrust is $\frac{1}{2}$ 1790 × (51.75 − 3.46) = 43,200 lbs.

When $i = \varphi$, and especially when $i > \varphi$, the thrusts by the two methods disagree. In Fig. 61 where $i > \varphi$, it is seen that

AS departs considerably from a straight line and especially for the depth $x_o = BC$, the thrusts by the two methods differ quite appreciably.

Thus for $i = 25°$, $\varphi = 20°$, $c = 200$ lb./ft.^{2}, $w = 100$ lb./ft.^{3}, we have $x' = 5.71$ ft., $x_o = 23.8$ ft. and the total active thrust on the vertical plane x_o in depth, by the circular diagram method is 14,150 lbs., Art. 94; whereas, by the construction of Fig. 68, it is found to be 12,600 lbs., the plane of rupture making an angle of about 37° with the horizontal.

With the same data, the total thrust on a vertical plane $x = 45.71$ ft. in depth, by the construction of Fig. 68 is 84,000 lbs., the plane of rupture making an angle of about 30° with the horizontal. The corresponding unit thrust r' at $x = 45.71$ (assuming a linear variation from $x = 5.71$, where it is zero) is found from the formula, $\frac{1}{2} r' \times 40 = 84{,}000 \therefore r' = 4200$ lb./ft.^{2}; whereas by the roughly approximate method of Art. 101, $r' = 4400$ lbs./ft^2.

103. Center of Pressure. The center of pressure on a wall with the inner face *AB*, Fig. 68, battered, is indeterminate. If it is assumed to vary uniformly from zero at *F* to a maximum at *A*, it will lie on *AF* at one-third its length from *A*. This hypothesis may suffice for small batters; but where *AB* has a large batter, it is safer to proceed as follows: resolve the thrust on *AB* into vertical and horizontal components and assume the horizontal component to act on *AF* at 1/3 *AF* from *A* and the vertical component to act on *AB* at 1/3 *AB* from *A*. The vertical component is a function of the surface slope, the weight of *ABh* and the friction. The friction at any point is due to the normal pressure multiplied by *tan* φ', and is increasing uniformly over *BF* (due to the normal component of the weight of earth vertically over *BF*) and at a much more rapid rate over *AF*. Considering the three influences, it seems safe to regard the center of pressure of the vertical component to act on *AB* at 1/3 *AB* from *A*.

Where the wall leans toward the earth, or *AB* is to the right of *Ah*, the center of pressure of the whole thrust can be safely taken on *AF* at 1/3 *AF* from *A*.

EXAMPLE. Consider a wall of uniform thickness 5.18 ft., leaning toward the earth, of vertical height 38.2 ft. and a batter of 1/8 or 1½ inches to the foot, the masonry weighing 150 lb./ft.^{3}. Suppose the wall backed by level-topped earth weighing 112 lb./ft.^{3}, with $c = 500$ lb./ft.^{2} and $\varphi = 18° 30'$.

Taking the thrust against the wall as making the angle 18° 30′ with the normal, find by the construction of Fig. 68, the thrust to be 11,300 pounds. Regarding this thrust as acting on AF at $\frac{1}{3}$ AF above A and combining with the weight of wall, the center of pressure on the base (drawn at right angles to the faces) is found to pass 0.47 ft. to left of its center. If the wall has a footing course 7.22 ft. in width, flush with the rear face, the center of pressure on its base passes 0.53 ft. to right of its center.

The dimensions are those of a wall built by Brunel, as given by Résal,* at the entrance to the Mickleton tunnel, the earth being blue lias clay whose angle of repose, when humid, was only 18° 30′. The wall, however, had a projection on its back—not a counterfort—and a footing course and it stands although its thickness is less than 1/6 its height. It is seen above that its stability is assured if the coefficient of cohesion is 500 lb./ft.^{2} Although the weights of the projection of the footing beyond the front face and the noted projection at the rear face (which sustained the weight of the earth over it) were not directly introduced in the computation, they were indirectly considered in taking the weight of masonry at 150 lb./ft.^{3} in place of 146 lb./ft.^{3}, the computed value.

If we take c = 300 lb./ft.^{2}, the earth thrust is found to be 18,400 lbs. and the resultant on the base of the footing of width 7.22 ft., cuts the base 0.9 foot from its center or within the middle third, so that the wall is stable with c = 300 lb./ft.^{2}, which is not an unreasonable value for the clay considered.

104. Braced Trenches. As a trench with vertical sides is dug, it is assumed that the usual sheeting, rangers, and bracing are put in and that the bracing is always well keyed up, so as to exert an active pressure on the earth. The least pressure necessary for stability can be found from the construction, Fig. 68, if AB, the face of the trench is taken vertical, Bb_4 and Af, horizontal, so that n_1c_1, n_2c_2, . . ., are drawn horizontally to intersection with gs_1, gs_2, ... The longest of the corresponding lines nc, to scale, gives the thrust that must be exerted by the braces to keep the prism of rupture from descending. But, as hitherto pointed out, this least force that the braces must exert per linear foot of trench is exactly given by the formula,

$$E = \tfrac{1}{2} (x - x')^2 \tan^2 \left(45^\circ - \frac{\varphi}{2}\right),$$

where, $x' = \dfrac{2\,c}{w} \tan \left(45^\circ + \dfrac{\varphi}{2}\right)$.

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* *Poussée des Terres, Deuxième Partie,* p. 160.

As a numerical illustration, assume a trench 40 ft. deep, $\varphi = 33^\circ 41'$, $c = 100$, $w = 100$ $\therefore$ $x' = 3.74$ ft. and $x - x' = 36.26$ ft.; hence the least force the braces must exert per linear foot of trench is,

$$50\,(36.26)^2 \times 0.2866 = 18{,}844 \text{ lbs.}$$

From this value, knowing the width of trench and the spacing of the bracing, the size of the members can be computed.

It is very likely that lower values of φ and much higher values of c can be counted on for consolidated earth.

As to the next question of the center of pressure, it is evident from the method used in construction that the distribution of stresses cannot be the same as for a retaining wall. For the braced trench, the earth at first simply resists the pressure exerted by the braces when first put in and keyed up tight, particularly on that upper portion where the active earth thrust is nothing or very small. The digging is now continued for several additional feet without bracing, so that no pressure can be exerted on the unprotected portion. Then sheeting, rangers, and bracing are put in to the bottom of the trench so far dug and again the bracing is keyed up. The bottom braces at first take only the stress due to the keying up, but as the construction proceeds the braces take more and more of the active earth thrust, which increases with the depth. In the case of a retaining wall the active earth thrust is exerted on the whole area below the depth x', but for the braced trench, no pressure can be exerted on an unsheathed area and the thrust that would be exerted on this area for a retaining wall is carried entirely by the bracing above it. It can thus very well happen that the upper or middle braces receive more stress in the end than the lower braces. In fact, this was asserted to be generally true, for well-drained material, by many engineers in the valuable discussion on Mr. Meem's paper,* though others asserted that in wet or saturated ground, the lower braces were most severely stressed.

If we divide the value of E above by the depth of trench to get the average intensity, it would seem to be safe to use double this to figure the intensity of stress at any point, and to design all the braces for double the average intensity, since the exact distribution of stresses is unknown.

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* "The Bracing of Trenches and Tunnels, with Practical Formulas for Earth Pressures," *Trans. Am. Soc. C. E.*, vol. LX., p. 1.

As shown above, the true value of E is less than that given above when no vertical cracks appear in the earth, which explains why constructors deem it so essential to get in the bracing, well keyed up, as soon as possible to prevent cracks, which are otherwise apt to form, particularly after heavy rains.

105. Surcharged Walls. When the surcharge is of the type shown in Fig. 69, lines $F2$ and 24 are drawn x' ft. below and parallel to Bb_2, b_2b_4, respectively. The weights of the trial prisms of rupture, $A1\,b_1\,B$, $A2\,b_2\,B$, $A3\,b_3\,b_2\,B$, etc., are found and laid off to scale from g, Fig. 68, and the construction proceeds as before to find the thrust against the wall AB.

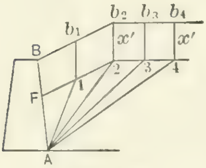

FIG. 69

As in the corresponding case of the surcharged wall, Art. 43, only with cohesion omitted, the center of pressure on AB, Fig. 69, probably lies above the point on AF, $1/3\ AF$ above A, particularly for high and steep surcharges. If this center is taken 0.4 AF above A, it would appear to be on the safe side. As the slope of Bb_2 diminishes toward zero, the factor changes gradually from 0.4 to $1/3$.

106. Pressures on Tunnel Linings. In Fig. 70 is shown a vertical transverse section of a tunnel of width b, length l, with the earth extending to a height h above it. If this tunnel has been driven by the use of shield or poling boards, the earth will tend to settle over it, and part of the weight of this earth directly over the tunnel will be sustained by the cohesion and friction (resulting from the lateral thrust) exerted along the sides and acting vertically upwards.

At the depth y below the surface, consider the conditions of equilibrium of a horizontal lamina of depth dy, width b, and length l.

Let V = the vertical unit pressure at depth y.
L = the horizontal unit pressure at depth y.

w = weight of earth per cubic unit.
φ = angle of friction, $\mu = \tan \varphi$.
c = coefficient of cohesion.
$A = b\,l$ = area of a horizontal section of tunnel.
U = perimeter of section as defined below.

Thus, when the ends as well as the sides are considered as offering support to the earth over the tunnel, $U = 2\,b + 2\,l$; for a very long tunnel of length l, let $U = 2\,l$; but for a short section under construction, for which $l = \frac{b}{2}$, say, the two sides and one (the front) end alone will be considered as offering a support, so that $U = 2\,l + b = 2\,b = 4\,l$. The first supposition, $U = 2\,(b + l)$, is identical with that pertaining to the ordinary rectangular bin. In fact the theory that follows is only a slight modification of Janssen's bin theory.

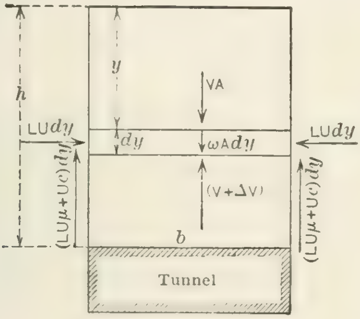

FIG. 70

For any one of the three cases, the total lateral force acting on the lamina is $L\,U\,d\,y$ and the frictional and cohesive forces acting vertically upwards on the sides and ends are $L\,U\,dy\,\mu$ and $U\,dy\,c$, as marked on the figure. Hence, if $V + \Delta V$ is the unit pressure on the under side of the lamina, acting vertically upwards, we have for equilibrium

$$(V + \Delta V)\,A - V A - wAdy + (L\,U\mu + Uc)\,dy = 0$$

One approximation has been introduced here, since for a depth $y' = \frac{2\,c}{w} \tan\left(45^\circ + \frac{\varphi}{2}\right)$, L is zero for cohesive earth, so that the term $(L\,U\mu + Uc)\,dy$ reduces to $Uc\,dy$, for this entire depth from the surface; but for large values of y the difference in final results will be inappreciable. Another approximation consists in placing,

$$L = Vk$$

where k is a constant to be determined by experiment. Many experiments on pressures in bins filled with grain, show that k is not a constant, still the supposition leads to fairly good practical results.

On substituting $L = Vk$ in the equation above and solving for $d\,V$ (put for ΔV),

$$dV = \frac{(w\,A - c\,U - k\mu\,U\,V)\,dy}{A}$$

$$\therefore \frac{-k\mu\,U\,d\,V}{w\,A - c\,U - k\mu\,U\,V} = -\frac{k\mu\,U}{A}\,d\,y.$$

On integrating,

$$\log_e (w\,A - c\,U - k\mu\,U\,V) = -\frac{k\mu\,U}{A}\,y + C$$

When $y = 0$, $V = 0$ $\therefore$ $C = \log_e (w\,A - c\,U)$

$$\therefore \log_e \frac{w\,A - c\,U - k\mu\,U\,V}{w\,A - c\,U} = -\frac{k\mu\,U}{A}\,y.$$

Putting $y = h$, then V (hereafter) will denote the vertical unit pressure at depth h, or at the roof of the tunnel,

$$\frac{w\,A - c\,U - k\mu\,U\,V}{w\,A - c\,U} = e^{-\frac{k\mu\,U}{A}h}$$

where $e = 2.71828\ldots$, is the Napierian base.

$$\therefore V = \frac{1}{k\mu}\left[w\frac{A}{U} - c\right]\left(1 - e^{-k\mu\frac{U}{A}h}\right) \quad . \quad . \quad . \quad (1)$$

$$L = V\,k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The formulas (1) and (2), give the values of the vertical and lateral unit pressures V and L, at the top of the tunnel, at the depth h below the free surface. The earth over the tunnel (or grain in a bin) will probably arch itself and it can be conceived to form a series of arches (or domes) superposed on each other. If each arch has the vertical thickness dy, it will have the same volume and weight $Awdy$ as the horizontal lamina of Fig. 70, by a theorem of the calculus. Further, if the reactions of the sides on the lamina marked on Fig. 70, be assumed to be those of the corresponding arch, their resultant may be supposed to represent the thrust of the arch at the sides. Similarly for the dome. It is thus seen that the horizontal lamina, having the same reactions as the corresponding arch (or dome), can be substituted for it in stating the equilibrium equations above. The value of V given by (1) is the average value at depth h or at the top of the tunnel.

Experiments on grain bins lead to the inference that the pressure at the middle of the roof is greater than that near the sides, but no law of variation can be stated. Similarly the lateral unit pressure $L = kV$, at the level of the roof can be found, but the variation of the lateral pressure over the vertical sides of the tunnel lining cannot be formulated, since so much of the weight of the earth directly over the tunnel, has been transferred by a kind of arch action to the sides. It is scarcely necessary to remark that the formula does not give the lateral pressure on the tunnel lining, caused by the swelling of clays saturated with water. The results refer to ordinary ground, either dry or moderately humid or to such earths as do not swell. The case of saturated earth will be discussed later.

As a numerical application, consider a long tunnel, where the ends are supposed to offer no support to the earth directly over the tunnel, so that $U = 2l$. If the width of tunnel is 15 ft., then $\frac{A}{U} = \frac{bl}{2l} = \frac{b}{2} = 7.5$ ft.

Let $\varphi = 30^\circ \therefore \mu = \tan\varphi = 0.577$, and assume $w = 90$ lb./ft.^{3}, $c = 100$ lb./ft.^{2}. As to the value to assume for k, no satisfactory experiments can as yet be appealed to. For an unlimited mass of earth devoid of cohesion, with level surface, $L/V = \tan^2\left(45^\circ - \frac{\varphi}{2}\right)$, but this value does not agree generally with the experimental determinations on large bins. Thus, Jamieson found from experiments on a wooden bin, about 12 by 12 ft., filled with wheat ($\varphi = 28°$), $k = 0.60$, whereas, $tan^2 (45° - \varphi/2) = 0.361$. The experimental value is 10/6 of this, so that from lack of definite data, there will be assumed for this value of φ,

$$k = \frac{L}{V} = \frac{10}{6}\, tan^2 \left(45° - \frac{\varphi}{2}\right).$$

For $\varphi = 30°$, this gives $k = 0.555$ and $k\mu = 0.320$. Formula (1) thus reduces to,

$$V = 1790 \left(1 - \frac{1}{e^{0.0427\, h}}\right).$$

V is expressed in pounds per square foot, h in feet.*

The following table gives the values of V, corresponding to assumed values of h. The results are shown graphically in Fig. 71. By the assumption, for any V, $L = kV = 0.555\, V$.

h Ft.	V Lb./Ft.^{2}	h Ft.	V Lb./Ft.^{2}	h Ft.	V Lb./Ft.^{2}
10	621	40	1,466	70	1,700
20	1,022	50	1,580	80	1,730
30	1,292	60	1,650	100	1,765

For large values of h, $e^{-k\mu \frac{U}{A} h}$ is very small, and as h increases indefinitely, eq. (1) approaches the limit,

$$lim\ V = \frac{1}{k\mu}\left[w\frac{A}{U} - c\right] \quad . \quad . \quad . \quad . \quad (3)$$

which in the present example, amounts to 1790 lb./ft.^{2}, a value which differs but little from that at $h = 100$ ft. This limiting value of V can then be regarded as practically the maximum value of V for large values of h. The expression for this limiting value of V—call it V_m—can easily be obtained independently.

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* A short table of Napierian logarithms is a convenience in computation. Thus when $h = 30$ ft., $e^{0.0427h} = e^{1.281} = 3.60$, since from the Table, $1.281 = log_e\, 3.60$.

Thus in Fig. 70, considering the whole series of horizontal lamina, it is plain that the greatest value of V would be realized if the weight of any lamina was entirely sustained by the friction on the sides (induced by the lateral thrust) and the cohesion acting there, for then this same condition of affairs would exist for all the lower lamina, V_m being transmitted vertically downward, unchanged to the tunnel. Stating this condition in algebraic form,

$$A\,w\,dy = (L\,U\mu + Uc)\,dy.$$

Putting $L = V_m k$, this equation reduces to (3) above. The equation is almost exactly true for large values of y and is practically true for much smaller values, as may be seen from the results above or from Fig. 71.

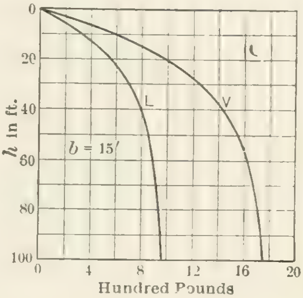

FIG. 71

Recurring to eq. (1), it is seen that when c is large enough, $\left[w\dfrac{A}{U} - c\right]$ may be zero or negative. In either case, it will be shown that the earth over the tunnel can be held up by the cohesion acting along the vertical sides, so that V must be placed equal to zero. Thus when U is the perimeter pertaining to the particular case, the total cohesive force that can be exerted for the height h is, $c\,Uh$. The weight of the vertical prism of earth directly over the tunnel is wAh, so that the cohesion alone can support the weight of the prism when $cUh \geqq wAh$ or when $c \geqq w\dfrac{A}{U}$, as stated.

From the three equations,

$$V_m = \frac{1}{k\mu}\left(w\frac{A}{U} - c\right),\quad k = \frac{10}{6}\,tan^2\left(45^\circ - \frac{\varphi}{2}\right),\quad L = k\,V_m,$$

the values of V_m and L, have been computed for $w = 90$ lb./ft.^{3} and various values of c, b, φ and A/U, and inserted in the following table:

c	b		V_m in Lb./Ft.²		L in Lb./Ft.²	
Lb./Ft.², (1)	Ft. (2)	φ (3)	$\frac{A}{U}=\frac{b}{4}$ (4)	$\frac{A}{U}=\frac{b}{2}$ (5)	$\frac{A}{U}=\frac{b}{4}$ (6)	$\frac{A}{U}=\frac{b}{2}$ (7)
100	15	30°	740	1,790	410	1,000
100	30	30	1,790	3,900	1,000	2,160
100	15	45	830	2,010	240	570
100	30	45	2,010	4,370	570	1,250
400	15	30	0	860	0	480
400	30	30	860	2,960	480	1,640
400	15	45	0	960	0	270
400	30	45	960	3,320	270	950

The pressures given in columns 5 and 7 refer to a long tunnel as previously explained. For a short length of tunnel about the heading, l was arbitrarily taken $= \frac{b}{2}$, and the earth directly over this length of tunnel was supposed partially supported by the friction and cohesion acting on the two sides and the front end, extended to the surface, whence $U = 2l + b = 2b$, $A = b^2/2$, therefore $A/U = b/4$. The corresponding pressures are given in columns (4) and (6).

The values for $c = 100$ are intended to apply to soft ground; those for $c = 400$, to hard consolidated earth. In both cases, the earth is supposed to be dry or moist but not saturated with water. For $c = 400$ lb./ft.², it is observed that the average pressures are both zero at the working faces of the 15 ft. tunnel. For very hard ground, there is but little doubt that c exceeds considerably 400 lb./ft.² or that the pressures will be smaller than any given. In fact only extended experimental determinations of the coefficient c and the angle of friction φ after the method suggested in Chap. I, can lead to satisfactory results in all cases where c and φ enter as determining factors. Mr. J. C. Meem, M. Am. Soc. C. E., in a paper on "Pressure, Resistance and Stability of Earth,"* gives a table of pressures on tunnel linings, which may be compared with the above.

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* *Trans. Am. Soc. C. E.*, vol. LXX, p. 387.

107. Saturated Earth. When the earth is completely saturated with water, the buoyant effort of the water must be considered, but there is by no means agreement as to how this shall be done. If the earth particles are to be treated as so many marbles, so that the water has free access to any horizontal or vertical plane, the procedure is simple. Thus, if a cubic foot of earth with 40% voids, weighs in air 90 lbs., it contains 0.6 cu. ft. of solids and the buoyant effort of the water equals the weight of an equal volume of water, or $0.6 \times 62.5 = 37.5$ lbs.; so that the weight of the cubic foot of earth in water is $90 - 37.5 = 52.5$ lbs. If for any earth, the weight of a cubic foot of the earth in water as determined in this way, is called w', then w' is to replace w in the formulas above and the constants c and φ to be used must be determined by experiment for the saturated earth.

Let h_0, Fig. 72, represent the depth from the water surface to the top of the tunnel. Then, when $h_0 = h$, on replacing w by w' and using the values of c and φ for saturated earth, formulas (1) and (2) will give V and L for the earth at the top of the tunnel. To these values add $62.5\,h_0$ to get the true pressures due to both earth and water. When $h_0 < h$, rough average values of w and w', c and φ may be used in getting an approximate value of the earth pressure. The values of w', c and φ used will evidently depend upon the ratio h_0/h.

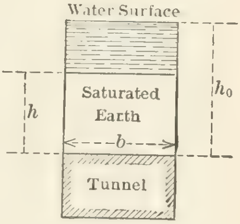

FIG. 72

When $h_0 > h$, lower and upper limits can be found as in the following numerical example, referring to a tunnel under a river through clayey silt, for which say $\varphi = 10°$, $c = 200$ lb./ft.^{2}, $w = 90$ lb./ft.^{3} $\therefore$ as above, $w' = 52.5$. Assume, $k = \tan^2$

$$\left(45° - \frac{\varphi}{2}\right) = 0.704 \therefore k\mu = k \tan 10° = 0.124.$$ Let $h = 40$ ft.,

$h_0 = 60$ ft. and for a long tunnel $b = 15$ ft. wide, $A/U = 7.5$.

Eq. (1), Art. 106, gives the vertical pressure, for $h = 40$ ft., due to the earth alone,

$$V = 1564\ (1 - e^{-.0166\,h}) = 1564 \times 0.485 = 760 \text{ lb./ft.}^2$$

This is less than the true pressure, as the weight of water above the earth must cause some additional pressure in the earth.

On adding the full water pressure for the height h_o, $62.5 \times 60 = 3750$, we find the total, 4510 lb./ft.^{2} Similarly we add 3750 to $k\ V$ to find the lateral unit pressure. An upper limit for the pressure corresponds to h indefinitely large, whence $V = V_m = 1564$. To this add the water pressure, 3750, giving the total 5314 lb./ft.^{2} The true vertical unit pressure thus lies between 4510 and 5314 lb./ft.^{2} and the true lateral pressure between $(3750 + k\ V)$ and $(3750 + k\ V_m)$ lb./ft.^{2}

For small values of φ, it seems more reasonable to assume $k = tan^2 \left(45^\circ - \frac{\varphi}{2}\right)$, in place of 10/6, this amount as before, since k is probably always less than unity In fact, for $\varphi = 0$ (water pressure) k is 1, so that whatever factor is applied to $tan^2 \left(45^\circ - \frac{\varphi}{2}\right)$ for the larger values of φ, it must gradually approach 1 as φ approaches zero.

The above solution may suffice for gravel, but for sand or silt with perhaps much clayey matter, the pores are more or less clogged up and there is perhaps intimate contact of a part of the earth with the roof of the tunnel, so that the water cannot get under it to produce a lifting effect, and the buoyant effort of the water is much diminished. Perhaps this may be more marked under small heads than large ones. Before any definite percentage of reduction can be stated, experiments on a large scale and on every kind of material usually met with, should be made; but as a mere numerical illustration, assume as found in Mr. Meem's experiment* that the water pressure through sand having 40% voids is diminished 40%.

With the data of the preceding example, the weight of the cubic foot of earth in water is now, $w'' = 90 - (0.4 \times 0.6 \times 62.5) = 75$ pounds. On replacing w by w'' in formula (1), we find $V = 1416$ lb./ft.^{2} To this add the water pressure, $0.4 (62.5 \times 60) = 1500$, and the lower limit to the vertical pressure at the top of the tunnel is found to be 2916 lb./ft.^{2} The upper limit is,

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* Art. 9, foot-note.

$$V_m + 1500 = 2918 + 1500 = 4418 \text{ lb. /ft.}^2$$

Similarly the lower and upper limits of L at the roof of the tunnel are $(1500 + k\ V)$ and $(1500 + k\ V_m)$ respectively.*

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* At first sight, it might be thought that a definite solution of the case where $h_o > h$, the weight of water over the earth being $62.5\ (h_o - h)$ lb./ft.^{2} = V' say, could be made by regarding the water as earth of equal weight and determining the constant C, Art. 106, by making $V = V'$ when $y = o$. The resulting value of V at the tunnel is greater than before by $V'\ e^{-k\mu \frac{U}{A} h}$; but if to this is added the remaining water pressure $62.5\ h$, the result will be found to be less than the lower limit in the first solution above. The reason is that the theory implies transference of a certain portion of the weight of water over the earth to the sides, which is inadmissible.

CHAPTER VI

THEORY PERTAINING TO BOTH DEEP AND SHALLOW BINS

108. A shallow bin may be defined as a bin where the surface of rupture cuts the free surface of the filling or contained material. Thus the ordinary hopper bin, containing coal, coke or ore, is usually a shallow bin.

A deep bin is one having a greater depth than the highest shallow bin for the same diameter, as in the case of grain bins. The theory pertaining to deep and shallow bins is entirely different, as may be surmised from what has preceded.

Grain Bins. The theory of Art. 106 applies to deep vertical bins filled with grain, on replacing μ by μ', the coefficient of friction of the grain on the wall and (for simplicity) making $c = 0$. The formulas for V and L, now take the simpler forms,

$$V = \frac{1}{k\mu'} \cdot w \frac{A}{U}\left(1 - e^{-k\mu' \frac{U}{A} h}\right),$$

$$L = k\,V = \frac{w}{\mu'} \cdot \frac{A}{U}\left(1 - e^{-k\mu' \frac{U}{A} h}\right).$$

For large values of h, we have practically, $V = \frac{w}{k\mu'} \cdot \frac{A}{U}$, $L = \frac{w}{\mu'} \cdot \frac{A}{U}$; so that L is independent of k. This is important, since for wheat alone, k has been found by different experimenters to vary from 0.3 to 0.67. For a rectangular bin, since all four vertical sides offer support, $\frac{A}{U} = \frac{b\,l}{2\,(b + l)}$, which reduces to $b/4$ for a square bin. For a circular section of diameter D, $A = \pi D^2/4$, $U = \pi D$ $\therefore$ $A/U = D/4$.

The stresses V and L are thus the same for a square bin and a circular one when $b = D$, or when the diameters are the same.

Janssen, who first deduced the above formulas for bins, found by experiments on bins of smooth cribbed boards,

$$k = 0.67, \ \mu' = 0.3 \ \therefore \ k\mu' = 0.20.$$

The bins contained wheat weighing 50 lb./ft.^{3}

Jamieson* in an elaborate series of experiments on full sized bins, found for wheat, $w = 50$, $\varphi = 28°$, $k = 0.60$, $\mu' = 0.44$, $k\mu' = 0.264$, the bin being of timber crib construction. A bin 12″ square and 6′ 6″ deep, of steel trough plate, was also filled successively with grain weighing in lb./ft.^{3}: for wheat 50, peas 50, corn 56 and flax-seed 41.5. It was found that the pressures in the case of the corn were about the same as for the wheat; for the peas, about 20% greater and for the flax-seed, 10 to 12% greater than in the case of the wheat.

It is possible that the values of k given above are too high, since experiments by Bovey, Lufft, Ketchum and Pleissner, give values of k varying from 0.3 to 0.6. Thus Pleissner gives for wheat in a small plank bin, $\mu' = 0.25$, $k = 0.34$, $k\mu' = 0.085$, and in a rough concrete bin, $\mu' = 0.71$, $k = 0.30$, $k\mu' = 0.213$. For a cribbed bin, k was found to vary from 0.4 to 0.5. The values of k for rye for the various cases, were roughly about ¼ less and the values of k' about ¼ greater. Jamieson found for wheat on steel flat plate riveted, $\mu' = 0.375$ to 0.4; on steel cylinder riveted, $\mu' = 0.365$ to 0.375; on concrete, smooth to rough, $\mu' = 0.4$ to 0.425 and on a cribbed wooden bin, $\mu' = 0.42$ to 0.45.

Some of the quantities given above have been taken from Ketchum's "Walls, Bins and Grain Elevators," to which the reader is referred for an exceptionally full treatment of the subject. It is found from the experiments that the lateral pressure of the grain on the bin walls increases very little after a depth of 3 times the width or diameter of the bin is attained, and that the pressures computed by the formulas above agree closely with the observed pressures. The curves of pressure shown in Fig. 71 are similar to those referring to grain bins. The pressures corresponding to grain running out of the discharge gates rarely exceeds by 10% those due to the grain at rest. The gates should be located near the center of the bin, for if located in the sides, the lateral pressure on the side opposite the gate is very materially increased.

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* *Trans. Can. Soc. C. E.*, vol. XVII, 1903; *Eng. News*, vol. 51, 1904, p. 236.

As an illustration of the application of the formulas, consider a square wooden cribbed bin, $b = 10'$ wide and $h = 70'$ high, filled with wheat weighing $w = 50$ lb./ft.³ Assume $k = 0.5$, $\mu' = 0.42 \therefore k\mu' = 0.21$. For the square bin, $A/U = b/4 = 10/4$ and for $h/b = 7$, it is sufficiently near to compute V from the formula,

$$V_m = \frac{w}{k\mu'}\frac{A}{U} = \frac{50 \times 10}{0.21 \times 4} = 600 \text{ lb./ft.}^2$$

The total pressure on the bottom is thus, $600\,A$ pounds, and since the weight of grain in the bin is $70 \times 50 \times A = 3500\,A$ pounds, the weight of grain carried by the sides through friction is $2900\,A$ pounds. Thus the sides of the bin carry $29/35$ or 71.5% of the whole weight of grain. This amount is nearly unvarying, even when the grain is running out, and incidentally, it is proof that the friction of the earth on a retaining wall is permanent, the conditions as to moisture being the same and that this friction is exerted at all times and not simply at the moment of failure, as is often contended.

The lateral pressure per square foot of bin wall, at the bottom, is, $kV_m = 0.5 \times 600 = 300$ pounds.

As another illustration, consider a smooth circular steel bin, 25 ft. diameter and 75 ft. high, filled with wheat. Assume, $w = 50$, $k = 0.5$, $\mu' = 0.37$ $\therefore k\mu' = 0.185$. We have $A/U = 25/4$; hence substituting in the formula for V above,

$$V = 1700\left(1 - \frac{1}{e^{2.2}}\right).$$

Since $e^{2.2} = 9.25$, $V = 1520$ lb./ft.²

The total pressure in pounds on a horizontal section 75 ft. below the surface of the grain is thus $1520\,A$; the weight of grain in the bin is $75 \times 50\,A = 3750\,A$; so that $2230\,A$ lbs. is carried by friction by the sides, where $A = \pi D^2/4$. On dividing this by the perimeter πD, the amount carried by a vertical strip 1 foot wide and 75 feet high is found to be, $2230 \times 25/4 = 13{,}925$ pounds. This amount added to the weight of the steel strip, is the shear at depth 75 ft. to be carried by the horizontal rivets for each linear foot of perimeter.

The lateral pressure at depth 75 ft., is $L = kV = 0.5 \times 1520 = 760$ lb./ft.²

Similarly the vertical and lateral pressures at any depth can be found from the formulas, by substituting for h the desired depth. Or preferably, y can replace h in the formulas, to give V and L at the variable depth y.

As noticed above, the value of k is the most uncertain quantity to be assumed and the numerical results just found will be somewhat altered if k is changed in value. Thus, if after Jamieson, we take $k = 0.6$, the other constants being unaltered, it is found at the depth $h = 75$ ft.,

$$V = 1256 \text{ lb./ft.}^2;\ L = 0.6\ V = 754 \text{ lb./ft.}^2$$

The weight of grain carried through friction to the vertical sides of the bin is now, (3570 − 1256) A lbs. and the weight per lineal foot of circumference is 15,590 lbs. This weight, transferred through friction, acts on the the inner side of the bin wall, and this eccentricity should be considered in designing any portion of the wall as a column, particularly in the case of thick wooden walls.

109. Shallow Bins. The case of the shallow hopper bin, filled with coal or other granular material, will alone be considered in what follows. The strict theory pertaining to the shallow bin is doubtless a very complicated one and it may be mentioned at the outset, that any approximate solutions indicated below, based upon a plane (and not a curved) surface of rupture, are open to objections, which will be clearly indicated. A simple solution for ordinary symmetrical bins will, however, be suggested that is believed to be on the side of safety.

To estimate the thrust on the sides of the bin, it is necessary to know the weight per cu. ft. (w) of the filling, its angle of friction (φ) and the angle of friction (φ') of the filling on the bin walls.

The following table gives values of w, φ and φ' for various materials:

Material	w Lb./Ft.^{3}	ϕ	ϕ'		
			Steel Plate	Wood Cribbed	Concrete
Bituminous coal. .	46–56	35°	18°	35°	35°
Anthracite coal. . .	47–58	27	16	25	27
Sand	90–	34–	18	30	30
Ashes	40–	40–45	31	40	40
Ore	125–	35–45	. .	. .	. .
Coke, piled loose .	23–32	. . . . .	25	40	40

There are three fundamental conditions to which the theory must conform:

(1) In the symmetrical bin, Fig. 73, symmetrically loaded, whether the filling is level at top or heaped, the thrust on the vertical plane CN, must be horizontal from considerations of symmetry.

(2) The thrust on the vertical plane AB will make the angle φ' with the normal to the plane and this direction does not change, exactly as experiment shows to be the case in deep bins.

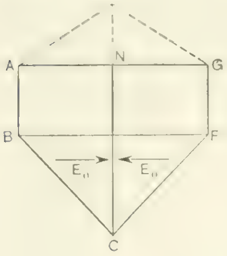

FIG. 73.

(3) The thrust on BC makes an angle with the normal to BC which cannot exceed φ' or equilibrium will be impossible.

110. Hopper Bin. Coal Heaped. The bin selected for computation is shown in cross-section, $ABCFGH$, Fig. 74, without the interior bracing. It is the same bin that figured in the rather extended discussion given in *Engineering News* in the years 1897-98.*

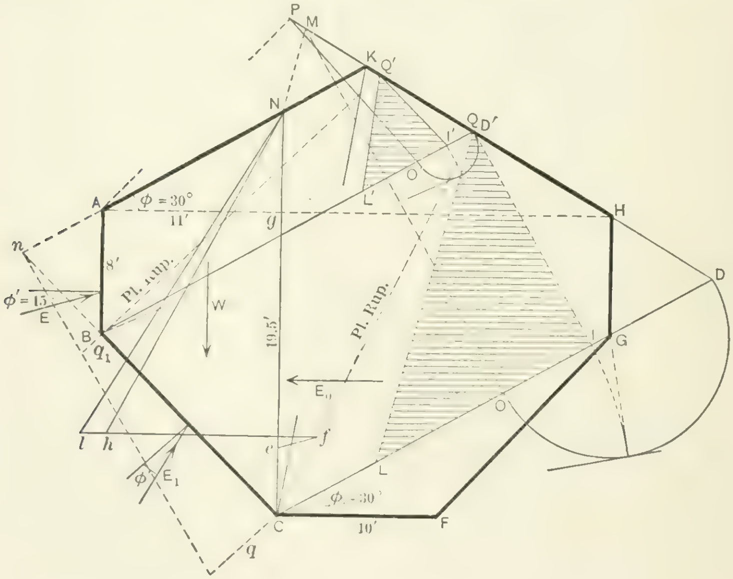

FIG. 74.

As in the original discussion, the values $w = 58$, $\varphi = 30°$, $\varphi' = 15°$ will be used below, though for anthracite coal, w

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* For a general discussion of bins, based on the author's formulas for earth pressure, see article by R. W. Dull in *Eng. News*, of July 21, 1904.

$= 52$, $\varphi = 27°$, $\varphi' = 16'$, are regarded as nearer the true values.

In this bin, $AB = 8'$, $Ag = 11'$, $Cg = 19.5'$, $CF = 10'$ and the pocket is $17'$ long. The influence of the friction at the ends in diminishing the thrust will be ignored. The analysis below refers, as usual, to 1 foot length of bin and the coal will be supposed heaped at the angle of repose, as shown in the figure.

It is easy to find the thrust E on the vertical plane AB, the thrust being inclined $\varphi' = 15°$ to its normal, by the graphical method, Art. 39, Fig. 19, for the surcharge AKH. Thus draw $AP \parallel BK$, to intersection P with HK produced. From P draw PO' making the angle, $\varphi + \varphi' = 45°$ with AB or with the vertical, to intersection O' with $BO'I'$, drawn parallel to AK or at the natural slope. Suppose BO' produced meets KH at D'. Using the mid-point of $O'D'$ as center, describe semicircle through O' and D' and lay off BI' equal to length of tangent from B to this semi-circle. From I', draw line parallel to PO' to intersection Q' with KH and lay off $I'L' = I'Q'$; then the thrust $E = w \times$ area shaded triangle $L'I'Q' = 58 \times \frac{1}{2}(6.4 \times 6.2) = 1150$ lbs.*

The plane of rupture is BQ'; whereas if the slope AK was unlimited, the plane of rupture would be BD' extended indefinitely and the thrust E would be found from the formula, Art. 44, to be, for $w = 58$, $h = 8$,

$$\frac{1}{2} w h^2 \frac{(\cos\varphi)^2}{\cos\varphi'} = 1440 \text{ lbs.}$$

The latter thrust acts $\frac{1}{3}AB$ above B and corresponds to a stress uniformly increasing from zero at A to 360 lb./ft.^{2} at B, whereas the true thrust $E = 1150$ lbs., is not uniformly increasing and acts about 0.35 to 0.36 AB above B, Art. 43. It will probably answer the purposes of computation to regard the resultant as acting at the third point and the stress on AB as uniformly increasing, from zero at A to $1150 \div 4 = 287$ lb./ft.^{2} at B.

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* In the case of a very acute intersection as at Q', the general graphical method of Art. 32 gives a closer result.

The method to follow in the case of the vertical side of the bin AB is clear and precise. The same cannot be said for the inclined side BC, either as to estimating the coal thrust on it or its point of application. Some assumption will have to be made and the simplest, for a *trial solution*, is to assume the thrust on the vertical plane CN as acting horizontally, so that $\varphi' = 0$ for this plane.

Hence, having drawn CD at the natural slope, to intersection D with KH produced, draw $NM \parallel CK$ to intersection M with HK produced; then draw MO making the angle $\varphi + \varphi' = \varphi = 30°$ with NC or with the vertical (whence $MO \perp CD$) to intersection O with CD. With the mid-point of OD as center, describe the semi-circle through O and D and lay off CI = length of tangent to semi-circle from C. Then draw from I a line parallel to MO to intersection Q with KH and lay off $IL = IQ$. It follows from Art. 39, that CQ is the plane of rupture and that the thrust is,

$$E_0 = \tfrac{1}{2}\, Q\, I^2 \times w = \tfrac{1}{2}\,(15.23)^2 \times 58 = 6730 \text{ lbs.}$$

This thrust will now be combined with the weight of coal $ABCN = W = 10{,}760$ lbs. and the reaction E of the side AB of the bin, to find the thrust on BC. Thus, lay off to scale, $Ne = W$, ef equal and parallel to E, and fh equal and parallel to E_0; whence Nh gives the amount and direction of the thrust on BC. In magnitude it is 11,900 lbs. and it is found to make an angle of 18° with the normal to BC.

This angle is very near the maximum allowable, $\varphi' = 15°$, so that the thrust is perhaps near the correct value. Referring now to the general conclusions of Art. 30 and Art. 32, since BC lies below the "limiting plane" and the construction above leads to a thrust on BC that makes a greater angle than φ' with its normal, it is next in order to assume E_1, the reaction of BC, as making the angle $\varphi' = 15°$ with the normal to BC and by a construction similar to that of Fig. 11, Art. 32, except that the reaction E is introduced, ascertain the magnitude of E_1.

This *second trial solution** is effected by drawing trial planes of rupture from C to points on KH, as Q, and regarding $ABCQK$, etc., as trial prisms of rupture, each subjected to four forces in equilibrium: its weight, the reaction E, the reaction E_1, and the reaction of the trial plane of rupture making the angle φ with its normal.

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* The construction is so similar to that in Fig. 77 or in Fig. 78, with the reactions of Ge_1, Ge_2, . . . , omitted, that it is not given here.

The weight is laid off vertically downwards to scale, say from N to e_1 (not shown), then e_1f_1 is drawn parallel and equal to E, then a line from f_1 parallel to assumed direction of E_1 to intersection c_1 with line from N parallel to the reaction of trial plane of rupture. Effecting similar constructions for the various trial prisms of rupture and reasoning as in Art. 21, it is seen that the greatest length of the type fc, gives to scale, the magnitude of the thrust on BC, $E_1 = 11{,}400$ lbs., which by assumption is inclined $15°$ to the normal to BC. It is found also, that the corresponding thrust on NC is inclined, having a horizontal component, acting to left, $= 7300$ lbs. and a vertical component acting upward $= 700$ pounds.

The result strictly refers to coal not restrained by the walls. FG, GH, on the right. Thus if we suppose $CF = 0$, so that the bin takes the shape Fig. 73, with the surface as per dotted lines, the construction just indicated gives a thrust on NC more inclined than before, whereas the restraint of the bin requires it to be horizontal. On that account, the result, although perhaps, a fair approximation, is not entirely satisfactory and since the construction demands a safe value, the following approximate construction is suggested:

After finding E_0 and E by the graphical method first indicated construct the force polygon $Nefh$. If it is found that Nh makes an angle with the normal to BC which does not exceed φ', then Nh can be regarded as the thrust on BC. If, however, this angle exceeds φ', then Nl is drawn making the angle φ' with the normal to BC to intersection l with fh extended and Nl (to scale) is to be taken as the thrust on BC. For the bin, Fig. 74, Nl (to scale) $= E_1 = 12{,}300$ lbs.

Strictly, the stress on BC is not uniformly varying, since the surface of the coal AKH is not a uniform slope, and the planes of rupture at different depths are not parallel. The law of uniform variation will, however, be assumed to derive formulas for the unit stresses at B and C, though it is admittedly approximate.

If $N A$ and $C B$ are extended to meet at n and any line is drawn through n, cutting perpendiculars to $B C$ at B and C, then the lengths of these perpendiculars, thus limited by the line, can represent the *magnitudes* of the (oblique) unit stresses q_1 and q at B and C; also the ordinate at any point of BC to the line will represent the magnitude of the stress there, according to the assumed law of uniform variation and the area of the trapezoid, having q and q_1 for parallel sides, is equal to E_1 in magnitude.*

For brevity, let $n C = l$, $n B = a$.

$$\therefore\ E_1 = \tfrac{1}{2}(q + q_1)(l - a);\ q_1 = q\frac{a}{l}.$$

On substituting the value of q_1 from the second equation in the first and solving for q, we obtain,

$$q = 2\, E_1 \frac{l}{l^2 - a^2}.$$

By measurement on the drawing, it is found that $l = 23.0'$, $a = 7.1'$ and since $E_1 = 12{,}300$ lbs., $\therefore\ q = 1180$ lb./ft.^{2}, $q_1 =$

$$1180 \times \frac{7.1}{23.0} = 370 \text{ lb./ft.}^2$$

These unit stresses at C and B act parallel to $N l$. The position of the resultant of all the stresses on BC is not altered if they are all supposed to act at right angles to $B C$; hence the distance y from C to where the resultant E_1 cuts $B C$, will be given by the formula of Art. 34, for the distance to the center of gravity of the trapezoid of which the two parallel sides are q and q_1, both supposed perpendicular to $B C$. We have,

$$y = \frac{h}{3}\,\frac{q + 2\,q_1}{q + q_1} = \frac{15.9}{3}\,\frac{1180 + 2 \times 370}{1180 + 370} = 6.6 \text{ ft.}$$

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* An error that is sometimes made, is to lay off q and q_1 parallel to E_1 and to regard the area of the trapezoid formed with q and q_1 as parallel sides, as the magnitude of E_1.

The pockets being 17 ft. long, the interior frames, shown in Fig. 75, are 17 ft. apart.

The total thrust on the vertical wall, AB, Fig. 74, for the 17 ft. length is $17 \times 1150 = 19{,}550$ lbs.; hence assuming $\frac{1}{3}$ of this carried to the framing at A and the remainder to B, we have the loads 6500 and 13,000 lbs. at A and B, both making the angle $\varphi' = 15^\circ$ with the normal to AB, to be used in estimating the stresses in the framing.

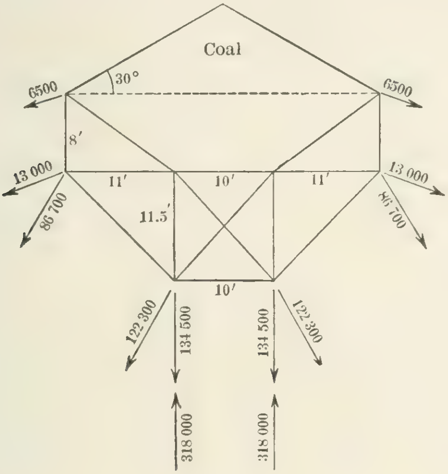

FIG. 75

Similarly, the total thrust for 17 ft. length of face BC is $17 \times E_1 = 17 \times 12{,}300 = 209{,}100$ lbs., whence the loads at B and C of the framing are,

$$209{,}100 \times \frac{6.6}{15.9} = 86{,}735 \text{ lbs.},$$

$$209{,}100 \times \frac{9.3}{15.9} = 122{,}265 \text{ lbs.},$$

both loads making the angle $\varphi' = 15^\circ$ with the normal to BC.

Finally, the weight of coal vertically over the bottom CF, for 17 ft. length of bin is $2 \times 134{,}500$ lbs., so that the loads corresponding carried to one frame, at C and F, are 134,500 lbs. each. The total weight of coal carried in one pocket is $2 \times 318,000$ lbs. and this should equal the sum of the vertical components of the loads at all the joints of the frame shown in Fig. 75. From these loads, the stresses in the framing can be ascertained, either by graphics or by taking moments.

Remark. A solution that has been proposed for this type of bin is as follows: extend BC and GF, Fig. 74, to intersection K' (not shown) and find by the construction, the horizontal thrust on KK' to be 8780 lbs.; combine this with the weight of nKK' to find the pressure on nK'; whence the unit pressure on nK' at K' can be computed, assuming the stress to vary uniformly from zero at n; whence the stresses q and q_1 at C and B can be found and the position and amount of E_1 determined as above explained.*

Assuming the thrusts on CN and BA to act horizontally, the true thrusts on those planes, by the construction of Fig. 74, are found to be 6728 and 1225 lbs. respectively. But corresponding to the law of uniformly varying stress on nK', the thrusts on the vertical planes KK', NC and AB must vary as the squares of the depths, which would give the thrust on $NC = 8780 \times \frac{NC^2}{KK'^2} = 5090$ lbs. and the thrust on $AB = 8780 \times \frac{AB^2}{KK'^2} = 487$ lbs., which are considerably less than the above

The proposed solution considerably underestimates the thrust on NC and should be rejected. A closer approximation is obtained by assuming the thrust on NC to be given by the construction, Fig. 74, its amount being 6728 lbs., whence the assumed horizontal thrust on AB, by the law of squares above, is $6728 \frac{AB^2}{NC^2} = 645$ lbs., which is about half the true thrust on AB, corresponding to $\varphi' = 0$. The influence of this thrust on the equilibrium polygon $Nefh$ is, however, small.

The force polygon is now drawn by laying off $Ne = W =$ weight of coal $ABCN$, then $ef = 645$ lbs., horizontally to right,

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* Ketchum's "Walls, Bins, and Grain Elevators," p. 192.

$fh = 6728$ lbs., horizontally to left, whence $Nh = 12{,}400$ lbs. = thrust on BC, which is practically the thrust, $Nl = 12{,}300$ lbs., found above. It makes an angle of about 17° with the normal to $B\,C$. The values of q, q_1 and y can be found by the equations above. Otherwise the thrust on NC, 6728 lbs., can be combined with the weight of coal $n\,NC = 11{,}960$ lbs., supposed to fill this space, to find the resultant on $n\,C = 13{,}700$ lbs. ∴ $\frac{1}{2}\,q\,.\,n\,C = \frac{1}{2}\,q \times 23.1 = 13{,}700$ ∴ $q = 1190$ lb./ft.² and

$q_1 = 1190 \times \dfrac{7.2}{23.1} = 371$ lb./ft.² Whence the thrust on $B\,C$ = $\frac{1}{2}\,(q + q_1) \times B\,C = 780.5 \times 15.9 = 12{,}410$ lbs., agreeing with the previous determination.

It is instructive to compare the results of the four approximate solutions given above, and they are grouped together in the following table:

Number	E_1	Angle E_1 Makes with Normal to BC	Thrust on NC	
			Horl. Compt.	Vert. Compt.
	Lbs.		Lbs.	Lbs.
1	11,900	18°	6,730	0
2	11,400	15	7,300	700
3	12,300	15	7,500	0
4	12,400	17	6,730	0

In (1), the thrust on NC is assumed to act horizontally. It is computed for the actual free surface and combined with E and W to find E_1 in direction and magnitude.

In (2), the obliquity of E_1 was assumed at $\varphi' = 15°$ and its magnitude found by a graphical method.

In (3), the horizontal thrust on NC was arbitrarily increased, so that E_1 should make the angle $\varphi' = 15°$ with the normal to $B\,C$.

The solution (4) is given under "Remark," Art. 110.

111. Hopper Bin. Coal Level at Top. The same bin and data, only with the coal level with the top, will be treated, (1) assuming, as before, that the thrust on the vertical plane Cg

is horizontal, Fig. 76. By the construction of Art. 40, we find E, the thrust per linear foot on wall AB, to be 560 lbs., the thrust being inclined $\varphi' = 15°$ to the horizontal. Similarly, the horizontal thrust on the vertical plane Cg, is found to be $E_o =$ 3700 lbs. and the weight of coal $ABCg = W =$ 8773 lbs.

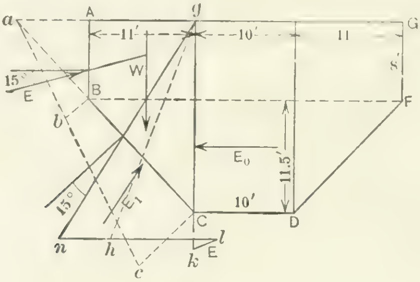

FIG. 76.

On laying off, to scale vertically, $gk = W$, kl equal and parallel to E, then horizontally to left from l, $lh = E_o$, then gh, the resultant on BC, measures 9180 lbs. and it is found to make an angle of 26° with the normal to BC, which exceeds the maximum obliquity allowable, $\varphi' = 15°$, by 11°.

(2) By the reasoning of Art. 29, it is next in order to assume that the unknown thrust on BC makes an angle of 15° with the normal to BC and by a construction similar to the "second trial solution" of Art. 110, to find the magnitude of this thrust. It is found to be 8100 lbs. The construction likewise shows that the active thrust on the vertical plane Cg has a horizontal component of 4800 lbs. and a vertical, upward component of 1700 lbs.; also that the result would be the same if CD, Fig. 76, was zero or for the bin of Fig. 73. But in the latter figure, the thrust on the medial plane, CX, is horizontal and not considerably inclined as above. The method then fails for this bin, Fig. 73, which must be attributed to the restraining influence of the walls in the right half of the bin and this throws doubt, though to a less extent, upon the construction to evaluate E_1 for the bin, Fig. 76. It may be seen also, that the upward

acting vertical component on Cg diminishes the value of E_1 materially as found by the first trial solution.

(3) As the case does not seem to admit of an exact solution, the following procedure is suggested: draw the line gn making the angle $\varphi' = 15°$ with the normal to BC to intersection n with the horizontal through l and regard gn as the thrust on BC. Its magnitude is $E_1 = 10{,}100$ lbs. This is decidedly on the side of safety, since the horizontal component $ln = 5800$ lbs. is greater than in either trial solution, so that a certain amount of passive thrust is involved.

(4) Turning now to the last solution indicated in " Remark," Art. 110 and regarding AB as a perfectly smooth wall, we have strictly, for the plane free surface AG, $E_1/E_0 = AB^2/Cg^2$, so that the stress on BC is uniformly increasing. In fact, if the wall AB is removed, a wall Ba added and the space BaA supposed filled with coal, the horizontal thrust on the vertical plane AB is the same as before and the magnitude of the stress on aC increases uniformly from zero at a to a maximum Cc at C.

On combining E_0 with weight of coal aCg it is found that the resultant on aC, makes an angle $26° +$ with the normal to aC. The magnitude of this resultant can be represented by the area of the right triangle aCc; whence unit stress at C = resultant divided by $\frac{1}{2}\,\overline{aC}$, whence the unit stress at B can be found and area trapezoid $BbCc$ = magnitude of $E_1 = 9380$ lbs. By the formula of Art. 110, it cuts BC at the distance y above C and its direction is that of the resultant on aC.

The results of the four approximate solutions are recorded in the following table:

Number	E_1	Angle Made by E_1 with Normal to BC	Thrust on Cg, Horl. Compt.	Thrust on Cg, Vert. Compt.
	Lbs.		Lbs.	Lbs.
1	9,180	26°	3,700	0
2	8,100	15	4,800	1,700
3	10,100	15	5,800	0
4	9,380	26	3,700	0

The values of E_1 differ considerably—the true value evidently lying between the extremes. For the value of E_1 selected, say No. 3, the quantities q, q_1 and y can be found from the formulas of Art. 110 and the solution completed exactly as in that article.

Presumably the bin will be designed for the "coal heaped," Fig. 74, where the agreement between the values for E_1 is much more satisfactory than for the case just treated.

112. Unsymmetrical Bin. Let us now consider the coal bin *ABCDFG*, Fig. 77, which only differs from the bin of Fig. 74 in the prolongation of the vertical wall *FG* to meet the surface of the coal, sloping at the angle of repose, $\varphi = 30°$ from *A* upwards to the right.

As before, assume $w = 58$ lb./ft.^{3}, $\varphi' = 15°$, so that the pressure on any side cannot make an angle with its normal that exceeds the angle of friction of coal on steel, $\varphi' = 15°$. The value of the thrust E_4 on *FG* is quickly found, by the construction of Fig. 20, Art. 40, to equal 4550 lbs. It acts $\frac{1}{3}$ *FG* above *F*. To find the thrust E_3 on the wall *DF*: by Art. 30, the first step would be to find the thrust on the vertical plane *DO* and then combine it with E_4 and W_1, the weight of coal *ODFG*, to find the thrust E_3 in amount and direction. This

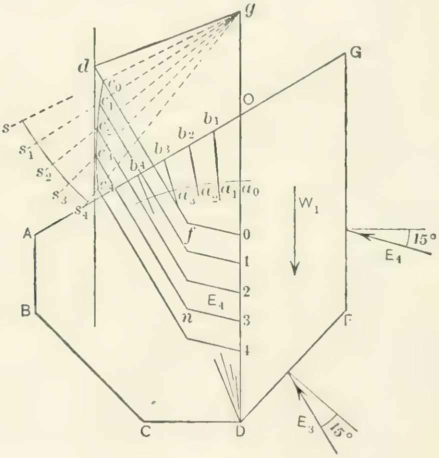

FIG. 77.

would be the true thrust on DF, unless its obliquity exceeded φ', in which case, the obliquity of E_3 is assumed equal to φ' and the magnitude of E_3 determined by the graphical construction given below. Unfortunately the direction of the thrust on DO is not known so that its magnitude cannot be found. In fact, when this direction is near that of AG, as in this case, a few degrees' change in this direction will make a large alteration in the magnitude of the thrust. Hence in this case, the first test cannot be applied and the second is alone available.

By this method, trial planes of rupture Db_1, Db_2, Db_3, . . ., are assumed and the weights of the prisms $GFDO$, $GFDb_1$, $GFDb_2$, . . ., computed and laid off, to scale, say from g, vertically downward to 0, 1, 2, etc. At these points, draw lines to the left equal and parallel to E_4, to points such as f and n. From such points, draw lines parallel to E_3 (assumed to make an angle of $\varphi' = 15°$ with the normal to DF) to intersections c_0, c_1, c_2, . . ., with the rays gs, gs_1, gs_2, . . ., which make the angles φ below the normals to planes DO, Db_1, Db_2, ... To construct these rays, with D and g as centers and a convenient radius $= Da_0 = gs$, describe arcs of circles $a_0 a_3$ and $s s_3$. Let DO, Db_1, Db_2, . . ., cut the first arc at a_0, a_1, a_2, . . ., and suppose gs drawn parallel to AG so that it makes the angle φ with the normal to the vertical plane DO. Then if we lay off $s s_1 = a_0 a_1$, $s s_2 = a_0 a_2$, . . ., the lines gs_1, gs_2, . . ., evidently make the angle φ with the normals to planes Db_1, Db_2, . . ., and thus give the required directions of the reactions of those planes—each taken in turn as a possible plane of rupture.

On drawing a vertical tangent to the curve through c_0, c_1, . . ., which happens to pass through c_3, the true thrust E_3 is given by the line nc_3 and the true (and only) plane of rupture is Ab_3. The proof is as follows: the prism $GFDb_3$ is in equilibrium under the action of four forces: its weight, E_4, E_3 and the reaction of Ab_3; represented by $g3$, $3n$, nc_3 and c_3g, the sides of a closed polygon. The line nc_3 is the greatest of the lines drawn parallel to E_3 and represents its magnitude; since n being fixed, if nc_3 is shortened, c_3 falls below its present position, so that the resulting reaction c_3g makes an angle greater than φ

with the normal to the plane Ab_3, which is inconsistent with equilibrium. Measuring nc_3 to the scale of loads, we find $E_3 = 14{,}300$ pounds.

On producing fc_0 to the vertical tangent to d, so that $fd = nc_3$, it is seen that dg represents the thrust on the vertical plane DO, since $GFDO$ is in equilibrium under its weight, the reactions E_4 and E_3 and the thrust on DO acting to the right, and these forces are proportional to the sides of the closed polygon $gofdg$. On measuring dg to the scale of *loads* it is found that, dg = thrust on plane $DO = 12{,}700$ lbs.

As a check, if the direction of the thrust on DO is assumed to be parallel to dg and the successive trial prisms of rupture are ODb_1, ODb_2, . . ., then either the general graphical method of Art. 21 or that of Art. 40, leads to the same thrust on DO of 12,700 lbs.

If the thrust on DO is assumed to act parallel to AG, its magnitude, for an unlimited mass of coal, would be, Art. 46,

$$\tfrac{1}{2}\, w \,.\, DO^2 \cos\varphi = \tfrac{1}{2}\,(58)\,(31.5)^2 \cos 30^\circ = 24{,}900 \text{ lbs.},$$

which is nearly twice the preceding value. The result is inapplicable to this problem, since it involves an indefinitely great prism of rupture on the right of DO, Art. 41, whereas the walls of the bin limit this prism.

The magnitude of E_3 having been determined, its point of application on DF can be found as in Art. 110, the point of intersection of DF and AG produced, corresponding to the point n of Fig. 74. Practically, for this bin, E_3 may be taken as acting at the mid-point of DF.

Thrust on BC. In Fig. 78 is given the construction for finding the thrust E_1 on BC. The possible prisms of rupture are $ABCG$, $ABCe_1G$, $ABCe_2G$, . . ., and their weights are laid off from a convenient point V, vertically downwards, to 0, 1, 2, . . ., respectively. By the construction of Art. 40, we find $E = 1440$ lbs. It acts $\tfrac{1}{3}\,AB$ above B.

At the points 0, 1, 2, . . ., draw lines equal and parallel to E (which is inclined 15° to the horizontal) to the right, to locate points such as f, b.

It is necessary now to compute the reactions, acting parallel to E_4, of the portions of the wall, Ge_1, Ge_2, . . ., which is easily done, since the thrust on $FG = E_4 = 4550$ lbs. and the thrust varies as the square of the depth. Thus the thrusts on Ge_1, Ge_2, . . ., are $\frac{Ge_1^2}{GF^2}E_4$, $\frac{Ge_2^2}{GF^2}E_4$, . . ., where $FG = 26.35$ ft., $Ge_1 = 5$ ft., $Ge_2 = 10$ ft., etc.

These thrusts on Ge_1, Ge_2, . . ., are to be laid off to the left, parallel to E_4, from the extremities of the lines drawn representing E. Thus at f, fg is drawn parallel to E_4 and equal to the reaction of Ge_2, etc.

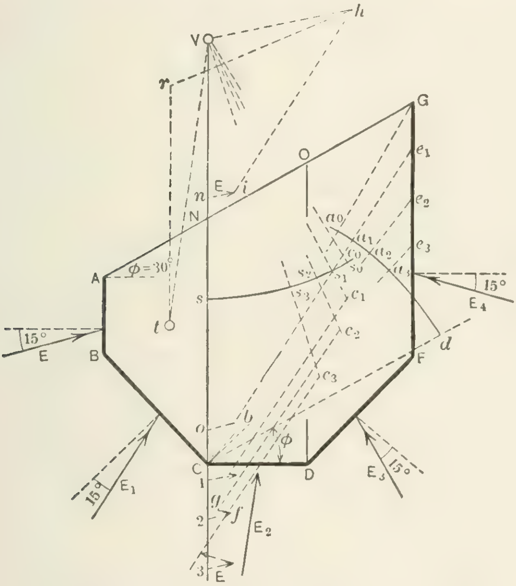

FIG. 78.

From the extremities of the last lines drawn, or from points such as g, lines are drawn parallel to the assumed direction of E_1 (which makes an angle of 15° with the normal to BC) to intersections c_0, c_1, c_2, . . ., with the rays from V inclined $\varphi°$ to the normals to planes CG, Ce_1, Ce_2, ... The directions of these rays are easily found, Art. 19, by first drawing Cd making the angle $\varphi = 30^\circ$ with the horizontal; then with C and V as centers and with the same radius, $Vs = C\,d$, describing arcs da_0, ss_0, and then laying off $ss_0 = da_0$, $ss_1 = da_1$, . . ., whence Vs_0, Vs_1, . . ., make angles φ with the normals to planes CG, Ce_1, . . ., as required.

The lines representing possible values of E_1, are those drawn parallel to E_1 and terminating in c_0, c_1, c_2, c_3. Of these lines, gc_2 is the greatest and represents the true thrust E_1. This is plain, if we note that $V\,2\,f\,g\,c_2\,V$ is a closed polygon, whose sides represent in turn, the weight of prism $A\,B\,C\,e_2\,G$, the reaction E of wall AB, the reaction (acting parallel to E_4) of part of wall Ge_2, the reaction E_1 of wall $B\,C$ and lastly, the reaction c_2V of plane $C\,e_2$. Similar closed polygons refer to the other trial prisms of rupture. Now as g is fixed, the reaction gc_2 cannot be diminished except by lowering c_2; but this would cause c_2V to make a greater angle than φ with the normal to plane Ab_2, which is inconsistent with equilibrium.

The line $g\,c_2$ thus represents E_1. Its value, to the scale of loads, is $E_1 = 14{,}600$ pounds. Its position on $B\,C$ is found by using the formulas of Art. 110.

The next step is to find *the thrust on the vertical plane NC*. To do this, lay off Vn vertically downwards, equal to weight of prism $A\,B\,C\,N$; then, $n\,i$ equal and parallel to E; next, $i\,h$ equal and parallel to gc_2, representing E_1; whence hV represents the coal thrust on CN in amount and direction. Its value is 9200 pounds.

It was ascertained above that the thrust on plane $D\,O$ was 12,800 lbs. Lay this off from h, Fig. 78, to the scale of loads used in this figure, parallel to its position gd given in Fig. 77, to point r and from r, lay off vertically downwards, $rt =$ weight of coal in prism $NCDO = 15{,}600$ lbs. Then Vt (to scale) $=$ 18,700 lbs. $= E_2 =$ pressure on bottom CD, acting parallel to Vt. This follows, because the prism of coal $NCDO$, is subjected to the thrust on $N\,C = Vh$, directed from left to right, a thrust on $D\,O = hr$, directed from right to left, its weight $= rt$ and the reaction E_2 of CD, which must equal tV to close the polygon $Vh\,r\,tV$, the sides of which represent the four forces in equilibrium. The point where E_2 cuts CD is uncertain—possibly it is near the center of CD.

The five reactions E, E_1, E_2, E_3, E_4, have now been computed. They refer to 1 ft. length of bin. If they are plotted in order, to scale, it will be found that a vertical line is necessary to close the polygon of length, to scale, equal to the total weight of coal, 45,800 lbs., thus checking the work. Of the above five reactions, E and E_4 are correct and E_2 depends to a small extent on the thrusts on the vertical planes CX and $D\,O$, which are in turn connected with the reactions on $B\,C$ and DF; hence the degree of reliability of the constructions for finding E_1 and E_3 is of practical interest.

With regard to E_1, the analysis is along similar lines to that for the bin, Fig. 74, for the coal heaped, in the "second trial solution," which led to fairly good results. We should expect no less in the determination of E_1 in Fig. 78, with the resulting construction for Vh, the thrust on XC. The latter thrust is nearly parallel to AG, which seems reasonable.

The construction for E_3, led to a thrust on the vertical plane $D\,O$ of 12,700 lbs., acting parallel to rh, Fig. 78. It may be thought that this thrust should be more nearly parallel to Vh, but it will be found that such a supposition leads to an obliquity for E_3 greater than 15° which is not admissible. Thus, if the direction of the thrust on $D\,O$ is assumed parallel to Vh, the value of the thrust is found (by the method of Art. 40) to be 9400 lbs. When this is combined with E_4 and the weight of $O\,D\,F\,G$, it is found to give a thrust on $D\,F$, 6% greater than before and making a greater angle than 15° with the normal to $D\,F$. Therefore, for such cases, by Art. 29, the construction of Fig. 77 is alone admissible and this leads to the value of E_3 given. From the various considerations above, it is thought that the values found above for E_1, E_2, E_3, are fairly good ones and that the constructions for finding them should prove serviceable to the constructor.

APPENDIX I

Stresses in Wedge-shaped Reinforced Concrete Beams

1. It is often necessary to find the stresses in reinforced concrete beams whose faces are inclined to each other, as in the case of the toes, heels, and counterforts of reinforced concrete retaining walls. Only an approximate solution can be offered for the general case, which, however, is on the side of safety and reduces to well-known results when the beam is taken as prismatic.

In Fig. 1 (1) is shown a portion of a toe, with an upper face making the angle β with the horizontal and subjected to a load Q on the lower horizontal face, acting vertically upward. The reinforcement, near and parallel to the lower face, is in tension. As usual, no tension will be supposed to exist along the vertical section IN, between the steel at I and the neutral axis O. On the portion ON, both compressive and shearing stresses are exerted.

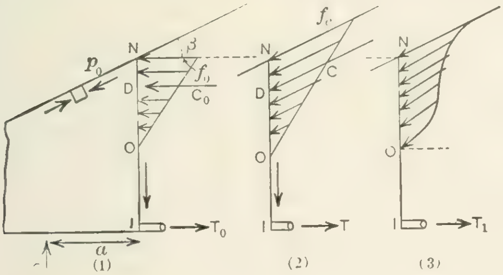

FIG. 1

The resultant unit stress at N is a principal compressive stress, acting parallel to the surface on a plane at right angles to it; for consider a small cube of concrete at N, with faces parallel and perpendicular to the upper surface. Since there is no shear on this surface (there being no matter above it to produce shear), it follows from a well-known theorem that there can be no shear on planes perpendicular to it; consequently, the unit stress p_o on a plane at N, at right angles to the surface, is normal to it; hence p_o is a *principal* compressive stress, and it acts parallel to the upper face.

2. It will next be shown that if f_o is the horizontal component of the unit stress at N on the plane IN, corresponding to the bending moment Qa, then,

$$p_o = f_o \sec^2 \beta.$$

To prove this important formula, consider the forces keeping the small prism ANB, Fig. 2, in equilibrium; the horizontal component of the unit stress on the vertical plane AB being f_o and the unit stress on the plane AN, perpendicular to the upper surface NB, being p_o. There is a shear on AB, but none on AN or NB. When AN is infinitesimal, the total stress on it is $p_o \,.\, AN = p_o \,.\, AB \cos \beta$, its horizontal component is $p_o \,.\, AB \cos^2 \beta$, and for equilibrium of ANB this must equal the total horizontal component of the stress on AB or $f_o \,.\, AB$.

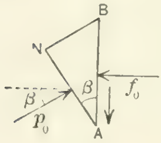

FIG. 2

On equating and solving, we derive, $p_o = \dfrac{f_o}{\cos^2 \beta} = f_o \sec^2 \beta$.

3. Let us suppose, as an approximation, that f_o is computed by the ordinary formula for a prismatic beam, just as if β was zero, so that f_o remains the same for any β. It would follow from the exact formula $p_o = f_o \sec^2 \beta$, as β approaches 90°, that p_o would become indefinitely large, which is absurd. The proposed approximation thus fails to give reasonable results for large values of β.

When $\beta = 0$, $p_o = f_o$ is an accepted value, but as β increases, it seems essential that f_o should decrease continually so that p_o shall remain within reasonable limits. The reason is evidently that the toe to the left of IN, Fig. 1 (1), is less resisting than a prismatic beam of the same length, and hence f_o for the toe is less than for the prismatic beam, and the conservation of plane sections after strain may not apply there.

It appears further that the excess in the computed value of f_o for the prismatic beam, over its true value for the toe, should be very small near the left or truncated end of the toe, since the resistances, for prismatic beam and toe, as regards bending, are now nearly equal, the common section IN being now near the end of the toe. As the section IN is taken farther from the end of the toe, the excess should increase and reach a maximum where the toe joins the stem.

The state of the horizontal stress on IN is then not uniformly increasing from O to N as indicated in Fig. 1 (1), but the true unit stress at N is less than as computed for a prismatic beam, the unit stress then probably increasing for a little distance below N and then decreasing as O is approached, since the sum of the moments of all the stresses on the section IN about I must equal the constant external moment Qa, whether the unit stresses are taken as uniformly increasing or as just indicated. These inferences all receive confirmation from the analysis of the experiments on india-rubber dams of Wilson and Gore and those on plasticine dams of Ottley and Brightmore,[*] a short discussion, pertaining to this subject, being given by the writer in *Trans. Am. Soc. C. E.*, vol. LXXVIII (1915), p. 747.

The assumption that f_o is to be computed as for a prismatic beam thus leads to an excess over the true stress at N, this excess being small, say for β less than 20°, but increasing more rapidly for larger values of β, so that the assumption, for lack of definite experimental data, will be limited arbitrarily to values of β less than 45°.

4. From analogy to the figures given by Morley[†] or Turneaure and Maurer,[‡] an attempt has been made to sketch in Fig. 1 (3) the principal compressive stresses at various points

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[*] *Minutes of Proceedings of Inst. C. E.*, vol. CLXXII, session 1907–08, Part *ii*, Pl. 3, Fig. 4, and Pl. 5, Figs. 19–21.

[†] "Strength of Materials," p. 138.

[‡] "Principles of Reinforced Concrete Construction," Art. 46.

of the section *IN*. A principal stress is to be defined as a stress which acts on a plane at right angles to its direction, there being no shearing stress on the plane. It thus acts normally to the plane and is the whole stress on it.

Besides the principal compressive stresses shown in Fig. 1 (3), there are principal tensile stresses, at right angles to the former, not drawn, but which must be included when the whole state of stress on *ON* is considered.*

5. Fig. 1 (2) shows a second approximate distribution of stress over the section *IN*, the stresses uniformly increasing from a point *O* and all acting parallel to the surface. In addition, there is a shearing stress on *IN*, which, added to the sum of the vertical components of the parallel stresses, must exactly equal *Q* or the shear due to the loading of whatever character.

The writer was led to make this distribution of stress because, as shown above, the stress at *N* acts parallel to the surface, and in the formulas that will be derived, this stress and the angle β are included from the start, so that the results promised to be more reliable than those pertaining to the state of stress Fig. 1 (1).

Before taking up the derivation of the formulas, it is well to show the relation between the state of stress assumed in Fig. 1 (2) and the true state represented by Fig. 1 (3), when the tensile principal stresses acting on *ON* are added. This relation can be easily shown in a manner kindly given by letter by Prof. H. A. Thomas,† involving the use of the ellipse of stress. Thus in Fig. 3, at a given point *P* on *ON*, let c = principal compressive unit stress and t = principal tensile unit stress (at right angles to c); then from the ellipse of stress, drawn with c and t as semi-axes, the resultant unit stress r on the plane *IN* at *P* can be found. Otherwise, Rankine's well-known construction for finding r can be used. This resultant r can be decomposed into a component p parallel to the compressed surface and a vertical component q.

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* Prof. O. H. Basquin has kindly sent the writer what he styles a "Rough Test of a Wedge-Shaped Glass Beam," with polariscope, to show the inclination of the principal compressive stresses to the horizontal. It is interesting to observe from the results that for about one-fourth of the way down this inclination was practically constant and equal to $\beta = 15.5°$ for the beam experimented with.

† Rose Polytechnic Institute, Terre Haute, Ind.

A similar construction being supposed, effected at every point of ON, the fundamental assumption made by the writer

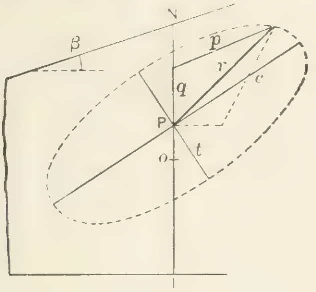


FIG. 3

can be thus stated: the component of the resultant stress r, parallel to the surface in compression, at any point of the section IN, is assumed to increase directly as the distance of the point from the neutral axis. The neutral axis may be defined as the line where the parallel component vanishes.

This is in agreement with the basis for the common theory of flexure for prismatic beams, which may be thus stated: the normal component of the resultant stress on the reference section at any point is assumed to increase directly as the distance of the point from the neutral axis. This use by Prof. Thomas of the ellipse of stress furnishes a scientific basis for the system of parallel stresses proposed.

6. It is seen that where the construction is supposed, made for each point of the section from O to N, the result will be not only the system of parallel stresses, Fig. 1 (2), but in addition a vertical shear, acting downward along NO, equal to Σq, or to the sum of the vertical components q above referred to. The total vertical shear on ON is thus $\Sigma q +$ the sum of the vertical components of the parallel stresses.

The point O in Fig. 1 (2) is to be determined by a formula derived further on. If the point so determined is identical with the true neutral point O of Fig. 1 (3), then the two systems of stresses Figs. 1 (2) and (3) [with the tensile principal stresses added] are equivalent and the moment of either system about I is the same.

However, as noted above, the parallel stresses cannot be regarded strictly as varying uniformly for a wedge-shaped beam, so that the determination of O on that basis can be regarded as only approximative—sufficiently near for the smaller values of β, but departing more and more from accuracy as β increases. Until direct experiment can speak definitely on this subject, it is suggested that the applications of the theory that follows be limited to angles of β not exceeding 45°.

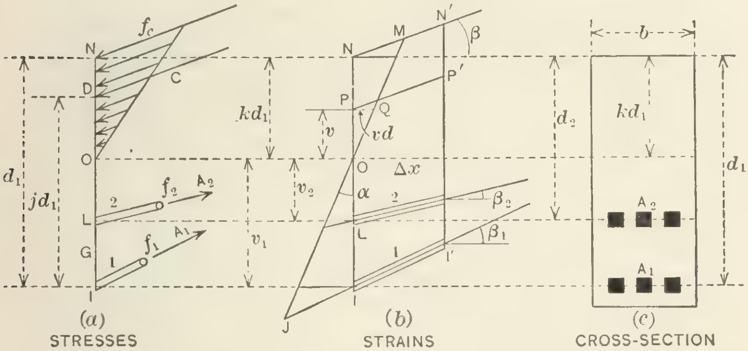
The stress p in Fig. 3 is an oblique stress per unit of area of the plane IN. Thus if we consider a cylindrical fiber, parallel to the compressed surface whose section in the plane IN has an area Δa, the total stress on the area is $p\ \Delta a$. As it is desirable to refer the stress to a plane perpendicular to p or to the fiber, let f equal the normal unit stress on such a plane. The area of a section of the fiber at right angles to its length is $\Delta\, a\ cos\ \beta$ and the total stress on it is $f\ \Delta a\ cos\ \beta$. Since this must equal $p\ .\ \Delta a$, we find the relation $f = p\ sec\ \beta$; hence f varies with p and thus follows the linear law of variation. In Fig. 1 (2), any one of the parallel unit stresses may be designated by the letter f and the unit stress at the surface, which acts on a plane perpendicular to it, will be designated by the letter f_c. As shown above, this is a principal stress.

The previous discussion refers to the most difficult case: that where the compressed surface is not at right angles to the section IN, which is parallel to the loads. If in Fig. 1, the load Q acts down, then the reinforcement must be placed near the upper inclined surface and the lower horizontal surface will be in compression. By the same assumption of parallel stresses, the compressive stresses on IO will be supposed to act normally to IN or parallel to the lower surface, as in the common theory.

7. A general solution will now be given which includes every

possible case. All special cases can be at once derived from this general solution

The usual hypotheses, that no tension exists in the concrete on the reinforced side of the neutral axis and that plane sections. remain such after stress, will be adopted.



(a) STRESSES (b) STRAINS (c) CROSS-SECTION

FIG. 4

In (*a*) and (*b*), Fig. 4, are shown two longitudinal sections, and in (*c*) a section of a part of a beam subjected to stress. *IN* represents a section of the beam taken always parallel to the direction of the loads, which may be weights, soil reactions, earth thrusts, etc. The shear due to the loads thus acts along *IN*, and the moment is the same for any point of the section.

For the breadth, b, let A_1, A_2, . . ., represent the areas of the cross-sections (taken at right angles to the axes) of the bars, 1, 2, . . ., at depths d_1, d_2, . . ., respectively.

Let f_1, f_2, . . ., represent the unit stresses in the bars, 1, 2, . . ., so that the total stresses in the successive layers of bars, 1, 2, . . ., for the breadth, b, are $f_1 A_1$, $f_2 A_2$, . . ., respectively.

Note carefully that the areas A_1, A_2, . . ., are *not* of sections in the plane *IN* but of right sections of the bars. Also f_1 is the unit stress on a right section of bar 1, so that $f_1 A_1$ is the stress in the bar. Similarly for the others.

The angles, β, β_1, β_2, . . . (expressed in radians), are those made by the surface, *NN'*, bars 1, bars 2, . . ., respectively, with the normal to the section, *IN*.

Let O be the neutral axis, and D the point where the resultant C of the compressive forces (all acting parallel to NN'), meets IN.

Let $NO = kd_1$, $DI = jd_1$, $OI = v_1$, $OL = v_2$, etc.

$I'N'$ represents a section parallel to IN, Fig. 4 (*b*), and at a perpendicular distance, Δx, from it. The "fiber," PP', parallel to NN', of concrete, will be supposed to have a section made by the plane IN, whose area is equal to Δa; and the distance, OP, will be called v. Thus the area of a right section of the fiber is $\Delta a \cos \beta$; and, if f *is the unit stress on a right section of the fiber*, PP', the total compressive stress on the fiber will be $(f \Delta a \cos \beta)$.

After strain, suppose the plane section, IN, rotates relatively to $I'N'$ through the angle, α, to JM. The section, JM, will thus be supposed to be plane, as in the ordinary theory.

For any fiber, whether of concrete or steel,

$$\text{unit stress} = \frac{\text{change of length of fiber}}{\text{length of fiber}} E \quad . \quad . \quad (1)$$

where, E = modulus of elasticity of fiber.

Let E_s = the modulus of elasticity of steel,

E_c = the modulus of elasticity of concrete,

and,

$$n = \frac{E_s}{E_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As in wedge-shaped beams, there is generally a stress at right angles to an interior fiber, the hypothesis of a constant E_c may not be exactly realized. However, any error from this source should be small.

Within working limits of stress, and for the very small values of α corresponding to very small values of Δx, the change of length of $PP' = PQ = v \alpha \sec \beta$, very nearly;* hence the units stress, f, on this fiber, $P\,P'$, by equation 1 is

---

* In the triangle, POQ, by the law of sines, $PQ = \frac{v \sin \alpha}{\cos (\alpha + \beta)}$. As α tends toward zero, this approaches $\frac{v \alpha}{\cos \beta} = v \alpha \sec \beta$.

$$f = \frac{P\,Q}{P\,P'}\,E_c = \frac{v\,\alpha\,\sec\beta}{\Delta\,x\,\sec\beta}\,E_c = \frac{v\,\alpha}{\Delta\,x}\,E_c.$$

Hence, as this unit stress on a right section of the fiber acts on the right sectional area, $\Delta\,a\,\cos\beta$, the total stress on the fiber is

$$\frac{v\,\alpha}{\Delta\,x}\,E_c\,(\Delta\,a\,\cos\beta),$$

and its component perpendicular to $I\,N$ is

$$\frac{E_c\,\alpha\,\cos^2\beta}{\Delta\,x}\,(v\,\Delta\,a).$$

The sum of such components on the compressed area of depth, $k\,d_1$ and breadth, b is,

$$\frac{E_c\,\alpha\,\cos^2\beta}{\Delta\,x}\sum_{o}^{kd_1}(v\,\Delta\,a) = \frac{E_c\,\alpha\,\cos^2\beta}{\Delta\,x}\left(\frac{1}{2}\,b\,k^2\,d_1^2\right),$$

since $\sum_{o}^{kd_1}(v\,\Delta\,a)$ = the statical moment of the area under compression about the neutral axis $= b\,k\,d_1\,\frac{1}{2}\,k\,d_1$.

For the layer of bars 1, at I, similarly, the unit stress is

Otherwise in Fig. 5, $R\,Q = O\,R\,\tan\alpha \therefore P\,Q = R\,Q\,\sec\beta = O\,R\,\tan\alpha\,\sec\beta$. From the figure, β being constant, as α approaches zero indefinitely,

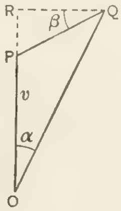


FIG. 5

Q approaches P, $O\,R$ approaches $O\,P$ and $\tan\,\alpha$ approaches α; whence from the formula $P\,Q$ approaches indefinitely,

$$O\,P.\,\alpha\,\sec\beta = v\,\alpha\,\sec\beta.$$

$$f_1 = \frac{JI}{II'}\, E_s = \frac{v_1\, \alpha\, sec\, \beta_1}{\Delta\, x\;\, sec\, \beta_1}\, E_s = \frac{E_c\, \alpha}{\Delta\, x}\,(n\, v_1).$$

Also, as the area of a right section of bars 1, for the breadth, b, is A_1, the total stress in the bars 1, is

$$\frac{E_c\, \alpha}{\Delta\, x}\,(n\, v_1\, A_1).$$

Similarly, the stresses in bars 2, 3, . . ., are

$$\frac{E_c\, \alpha}{\Delta\, x}\,(n\, v_2\, A_2),\; \frac{E_c\, \alpha}{\Delta\, x}\,(n\, v_3\, A_3),\; . \;.\; .$$

As all the loads on the beam were supposed to act parallel to $I\,N$, the part of the beam to the left of this section is in equilibrium under the loads and reactions acting on it and the internal stresses along $I\,N$. For equilibrium, the sum of the components of the stresses perpendicular to $I\,N$ must be zero. Therefore,

$$\frac{E_c\, \alpha}{\Delta\, x}\, cos^2\, \beta\, \left(\frac{1}{2}\, b\, k^2\, d_1^2\right)$$

$$= \frac{E_c\, \alpha}{\Delta\, x}\, n\, (v_1\, A_1\, cos\, \beta_1 + v_2\, A_2\, cos\, \beta_2 + .\;.\;.).$$

From Fig. 4,

$$v_1 = d_1 - kd_1,\; v_2 = d_2 - kd_1,\; .\;.\;.;$$

hence, on substituting these values, striking out the common factor, and reducing, we derive,

$$\frac{1}{2}\, cos^2\, \beta\, b\, d_1^2\, k^2 + n\, d_1\, (A_1\, cos\, \beta_1 + A_2\, cos\, \beta_2 + .\;.\;.)\, k$$

$$= n\, (d_1\, A_1\, cos\, \beta_1 + d_2\, A_2\, cos\, \beta_2 + .\;.\;.) \quad .\quad .\quad (3)$$

From this quadratic in k, the value of k is computed, and thus $k\, d_1 = O\,N$ can be found and the neutral axis located. Also, as the compressive stresses are uniformly varying, $D\,N = \frac{1}{3}\, O\,N$; therefore, $j\, d_1 = d_1 - \frac{1}{3}\, k\, d_1$; and

$$j = 1 - \frac{1}{3} k \quad . \quad . \quad . \quad . \quad . \quad (4)$$

8. *The Resisting Moment, M_s, of the Steel.* The moment, M, at the section, IN, due to the external forces, is equal to the resisting moment of the stresses acting along the section. Calling the perpendicular distances from D to bars 1, 2, . . ., p_1, p_2, . . ., respectively,

$$M_s = f_1 A_1 p_1 + f_2 A_2 p_2 + \dots \quad . \quad . \quad . \quad (5)$$

where, $p_1 = j\, d_1 \cos \beta_1$, $p_2 = D L \cos \beta_2$, etc.

Now, as $f_1 = \frac{E_c \alpha}{\Delta x} n v_1$, $f_2 = \frac{E_c \alpha}{\Delta x} n v_2$, . . ., it follows that

$$f_2 = \frac{v_2}{v_1} f_1,\; f_3 = \frac{v_3}{v_1} f_1, \dots \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or the unit stresses in the bars vary directly with the distances from the neutral axis. The unit stresses in the interior bars will thus always be less than f_1, so that such an arrangement of bars is uneconomical.

On substituting equations 6 in equation 5,

$$M_s = f_1 \left(A_1 p_1 + \frac{v_2}{v_1} A_2 p_2 + \frac{v_3}{v_1} A_3 p_3 + \dots \right) . \quad . \quad (7)$$

If preferred, after locating the point D on a drawing, the perpendiculars, p_1, p_2, . . ., can be measured to scale; otherwise, they may be computed readily by the formulas given.

If the resisting moment of the steel, M_s, is less than that of the concrete, M_c, for assigned maximum unit stresses, then the moment, M, of the external forces is put equal to the right member of equation 7, the value of f_1 ascertained, and, from equation 6, the values of f_2, f_3, . . ., are computed. The stresses in bars 1, 2, . . ., are thus $f_1 A_1$, $f_2 A_2$, . .

Otherwise, if a certain value is assigned to f_1, as 16,000 lb. per sq. in., and A_2, A_3, . . ., are assumed, from equation 7, A_1 can be computed. For rough computations, A_2. A_3, . . ., may often be ignored, in which case we can write,

$$M_s = f_1 A_1 p_1 = f_1 A_1 j d_1 \cos \beta_1 .$$

9. *The Resisting Moment, M_c, of the Concrete.* To compute M_c, the position of the resultant, R, of the stresses in the bars, must first be found. The magnitudes of the forces acting on the bars 1, 2, 3, . . ., are

$$f_1 A_1, f_1 \frac{v_2}{v_1} A_2, f_1 \frac{v_3}{v_1} A_3, \ . \ . \ .,$$

and, as these are proportional to f_1, the direction and line of action of R are the same for any value of f_1 and hence for $f_1 = 1$.

Let H = the component of R perpendicular to IN when $f_1 = 1$; then

$$H = A_1 \cos \beta_1 + \frac{v_2}{v_1} A_2 \cos \beta_2 + \ . \ . \ .$$

Suppose the resultant cuts IN at G; then, taking moments about D,

$$H \, . \, DG = A_1 p_1 + \frac{v_2}{v_1} A_2 p_2 + \frac{v_3}{v_1} A_3 p_3 + \ . \ . \ .$$

The right member, presumably, has already been computed in applying equation 7; hence DG is quickly ascertained and the point, G, located.

Call the maximum unit stress on the concrete at N, f_c; the unit stress on the fiber, PP', at P is thus $\frac{f_c}{kd_1} v$. This acts on the area ($\Delta a \cos \beta$); hence the stress on the fiber is

$$\frac{f_c}{kd_1} v \, \Delta \, a \cos \beta,$$

and the sum of such stresses is

$$C = \frac{f_c}{kd_1} \cos \beta \sum_{o}^{kd_1} (v \, \Delta \, a) = \frac{1}{2} \frac{f_c}{kd_1} \cos \beta \, b \, (kd_1)^2.$$

Therefore, $C = \frac{1}{2} f_c \, b \, kd_1 \cos \beta$.

Taking moments about G,

$$M_c = \frac{1}{2} f_c \, b \, kd_1 \cos^2 \beta \, \overline{DG} \quad . \quad . \quad . \quad . \quad (8)$$

If, for assigned maximum values of f_1 and f_c, $M_s > M_c$, the beam is over-reinforced and M is placed equal to the right member of equation 8, as its strength is now limited by that of the concrete. This case rarely occurs in practice.

10. As an illustration of the application of the formulas, take the counterfort, BM, Fig. 6, attached to the face-slab, NM, and suppose the counterfort to be subjected to a horizontal earth thrust of 123,750 lb., acting to the left and 9.06 ft. above N, giving a bending moment at N of 13,454,100 in.-lbs. The section, NB, corresponding to NI of Fig. 4, is taken parallel to the load (the earth thrust), and is therefore horizontal.

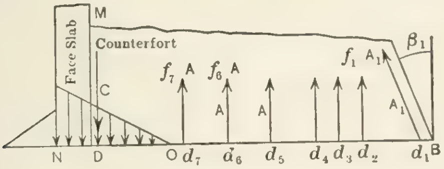


FIG. 6

The width of the counterfort is $b = 18$ in. The inclined bars have a total sectional area $= A_1 = 9.45$ sq. in., and the vertical bars, a common area, $A = 0.784$ sq. in. Using the foregoing notation, $\beta_1 = 23° 58'$, $\beta_2 = \beta_3 = \ldots = \beta_7 = 0$. The distances, d, were measured from N, or from the front face of the vertical slab; $d_1 = 128$, $d_2 = 108$, $d_3 = 100$, $d_4 = 92$, $d_5 = 76$, $d_6 = 60$, $d_7 = 44$, all in inches. Assume $n = 15$.

On substituting known values in equation 3, we derive $k = 0.311$. Therefore, $j = 1 - \frac{1}{3} k = 0.896$; whence $NO = kd_1 = 39.8$ in., $DB = jd_1 = 114.7$ in.; also, $ND = \frac{1}{3} NO = 13$ in.

On subtracting $NO = 40$ from d_1, d_2, . . ., we derive v_1, v_2, etc. The perpendiculars, p_1, p_2, . . ., from D on bars 1, 2, . . ., are $p_1 = jd_1 \cos \beta_1 = 105$, $p_2 = d_2 - ND = 95$, . . ., respectively.

From equation 7 we have, $13{,}454{,}100 = 1164\, f_1$; whence $f_1 = 11{,}500$ lb. per sq. in.; from equation 6, $f_2 = 8970, \ldots,$ $f_7 = 575$ lb. per sq. in.

As the weight of the heel-slab must be carried by the rods, the areas and spacing of the vertical rods were designed to carry their proportionate part of the weight of the heel-slab. The stresses corresponding are found to be in excess of those due to the moment, M. This excess is taken up by the bond stress in a short distance above NB, so that, above a certain level, only the moment stress corresponding to that level is carried by the vertical rods.

The total stress in the inclined rods, $f_1A_1 = 11{,}500 \times 9.45 = 108{,}675$ lb., is, of course, less than the stress, 127,000 lb., found by ignoring the influence of the vertical rods. This last stress is most easily found by use of the diagram, Fig. 9, and equation 11, given later.

This example of the counterfort has been given more for the purpose of gaining an idea of the actual stresses involved than of urging the adoption of the more refined method in practice.

If the diameter of a bar is to be some multiple of ⅛ in., there may be no saving by the use of the more exact method. It must be remembered, too, that the vertical bars are not always bonded securely in the base-slab, the earth thrust may also be much increased in times of heavy rains, where the filling is not adequately drained; and, further, the foundation may be more yielding than estimated.

11. In the next example, Fig. 7, representing the heel of a T-wall with a fillet, the wall being on the point of overturning, the exact method seems advisable.

The total moment at the section, NI, due to the two forces shown, is

$$M = (13{,}000 \times 2.5 + 5333 \times 5)\,12 = 709{,}992 \text{ in.-lb.}$$

The reinforcement, shown by the broken lines, for both inclined and horizontal bars, consists of 7/8-in. square bars, 8 in. from center to center, corresponding to $A_1 = A_2 = 1.15$ sq. in. for a breadth of $b = 12$ in.

Assuming $n = 15$, $\beta_1 = 43° 10'$, $\beta = \beta_2 = 0$, $d_1 = 42$, $d_2 = 21.5$, and substituting in equation 3,

$$\frac{1}{2} b\, d_1^2\, k^2 + n\, A_1\, d_2\, (\cos \beta_1 + 1)\, k = n\, A_1(d_1 \cos \beta_1 + d_2)$$

we find, after solution,

$$k = 0.238; \text{ therefore } j = 1 - \frac{1}{3} k = 0.921,$$

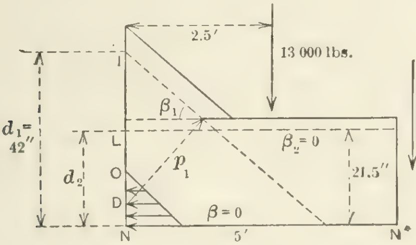


FIG. 7

$NO = kd_1 = 10$ in., $DI = jd_1 = 39$ in., $ND = 3.33$ in.,

$v_1 = d_1 - kd_1 = 32$, $v_2 = d_2 - kd_1 = 11.5$; therefore $\dfrac{v_2}{v_1} = 0.361$.

By equation 7, $M_s = f_1\, A_1 \left(p_1 + \dfrac{v_2}{v_1}\, p_2\right)$, $p_1 = DI \cos \beta_1 = 28.4$, $p_2 = 18.2$;

therefore $M = 710{,}000 = 1.15\, (18.2 + 6.57)\, f_1 = 40 f_1$.

therefore $f_1 = 17{,}750$ lb. per sq. in., $f_2 = \dfrac{v_2}{v_1} f_1 = 6380$ lb. per sq. in.

Where the foundation is good, there can be only a very small moving over of the wall, so that the friction force, 5333 lb., at N' can be neglected and the soil reaction included. In this particular example (not given in full here) the new values of f_1 and f_2, thus found, are only four-tenths of those given previously.

12. *Special Case. Where $\beta = 0$ and All the Rods Lie in One Plane Which Meets the Section, IN, in a Line at I.—Fig. 8.* The rectangular section has the breadth, b and, as before, A_1 represents the combined area of a right section of all the rods at I in the breadth, b. To agree with the usual notation, put $d_1 = d$ and $f_1 = f_s =$ the unit stress in the rods. Equation 3 now reduces to

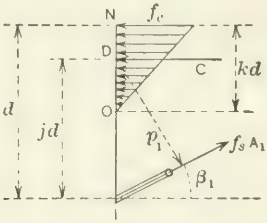


FIG. 8

$$\frac{1}{2} b\, d^2\, k^2 + nd\, A_1 \cos \beta_1\, k - nd\, A_1 \cos \beta_1 = 0.$$

Placing $p = \dfrac{\text{steel area}}{\text{concrete area}} = \dfrac{A_1}{bd}$, and dividing by $\dfrac{1}{2}\, bd^2$,

$$k^2 + 2\, n\, p \cos \beta_1\, k - 2\, n\, p \cos \beta_1 = 0,$$

therefore $k = -\, n\, p \cos \beta_1 + \sqrt{(n\, p \cos \beta_1)^2 + 2\, (n\, p \cos \beta_1)}$. (9)

When $\beta_1 = 0$, this reduces to the usual formula for prismatic beams,

$$k = -\, n\, p + \sqrt{(n\, p)^2 + 2\, (n\, p)} \quad . \quad . \quad . \quad (10)$$

In Fig. 9, the values of k and $j = 1 - \dfrac{1}{3} k$, are given as ordinates to the dotted curves for various percentages of steel and values of β_1, varying from 0° to 40°, assuming $n = 15$.

The resisting moment of the steel is found by taking moments about D,

$$M_s = f_s\, A_1 \cos \beta_1\, j\, d = f_s\, A_1\, p_1 \quad . \quad . \quad . \quad (11)$$

where $p_1 =$ the perpendicular distance from D on the bar.

The resisting moment of the concrete is found by taking moments about I,

$$M_c = \frac{1}{2} f_c\, b\, k\, d\, j\, d = \frac{1}{2} f_c\, (k\, j)\, b\, d^2 \quad . \quad . \quad (12)$$

For assigned maximum working values of f_s and f_c, the least resisting moment is equated to the moment of the external forces.

In using Fig. 9, note that p is not the percentage of steel,

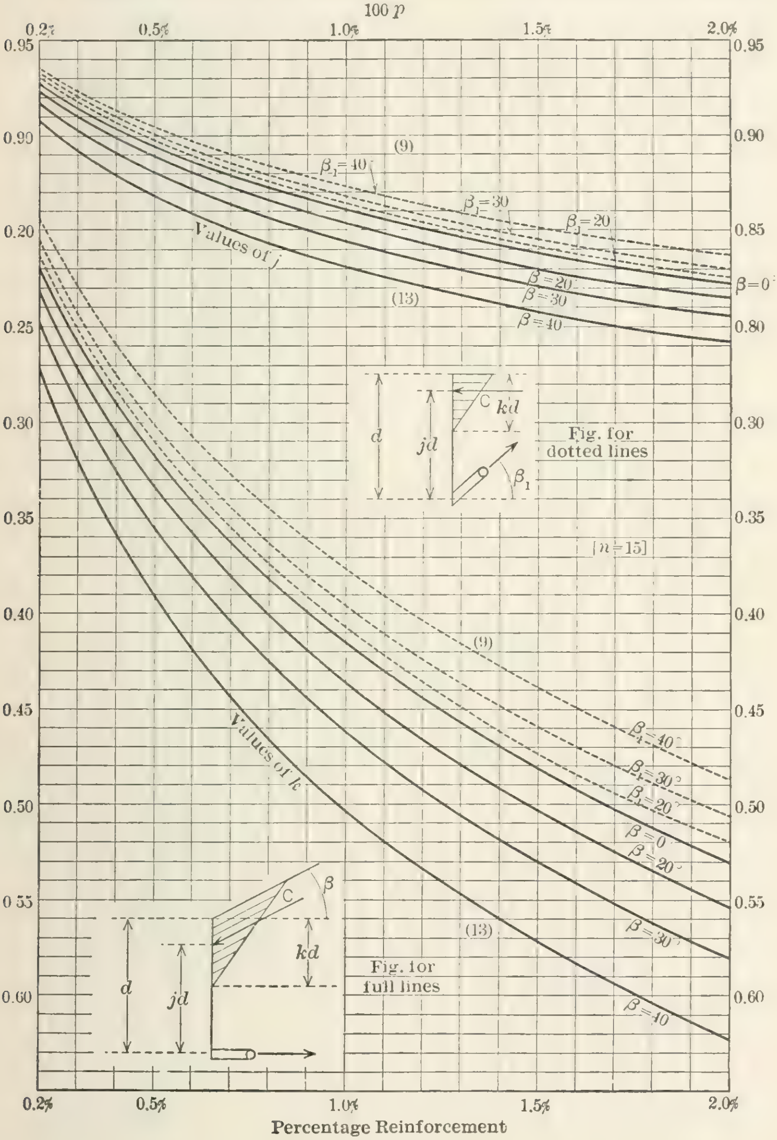


FIG. 9

but 1/100 of the percentage, the area of the steel A_1, being the area of a section of the rods at right angles to the axis and not of a section in the plane IN.

Resuming the example of the counterfort, Fig. 6, what will be the result of ignoring the vertical rods? With $A_1 = 9.45$ sq. in. and as the area of the base or section. $NB = 128 \times 18 =$ 2304 sq. in., $p = \frac{9.45}{2304} = 0.004$, or the steel percentage is 0.4. Using Fig. 9, with $\beta_1 = 24°$ and the steel percentage 0.4,

$$j = 0.906; \text{ therefore } j\,d = 0.906 \times 128 = 116.$$

Hence, as the moment of the earth thrust was given as $M =$ 13,454,100 in.-lb., by equation 11,

$$f_s\,A_1 \times 116 \times 0.914 = 13{,}454{,}100;$$

therefore $f_s\,A_1$ = the stress in the inclined rods = 127,000 lb. With $A_1 = 9.45$, as assumed, $f_s = 13{,}400$ lb. per sq. in., in place of 11,500 lb. per sq. in., found before, the vertical rods being included.

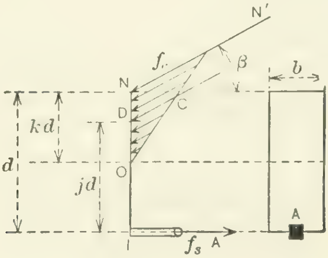


FIG. 10

13. *Special Case. Where the Rods Lie in One Plane, for Which* $\beta_1 = 0$, *but* β *is not Zero.*—Fig. 10. Putting d for d_1, and A for A_1, equation 3 reduces to

$$\cos^2 \beta\, bd\, k^2 + 2\,n\,A\,k - 2\,n\,A = 0.$$

Therefore, as before, putting $p = \frac{A}{b\,d}$,

$$k = \frac{1}{cos^2\,\beta}\left[-(n\,p) + \sqrt{(n\,p)^2 + 2\,(n\,p)\,cos^2\,\beta}\right] \quad . \quad (13)$$

In Fig. 9, the full lines give the values of k and $j = 1 - \frac{1}{3}k$, corresponding to $n = 15$ and to various values of β and percentages of steel. Equation 7, in this case, reduces to

$$M_s = f_s\,A\,jd \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

Also, as $D\,G = D\,I = jd$, equation 8 becomes

$$M_c = \frac{1}{2}f_c\,(kj)\,bd^2\,cos^2\,\beta \quad . \; . \; . \; . \; . \quad (15)$$

If, in Fig. 10, we write $p' = d\;cos\;\beta =$ the perpendicular from I on $N\,N'$ produced,

$$M_c = \frac{1}{2}f_c\,(k\,j)\,b\,p'^2 \quad . \; . \; . \; . \; . \; . \quad (16)$$

The product, kj, varies with β. In Table 1 the values of $(kj)\;cos^2\,\beta$, are given for various values of p and β. The values for $\beta = 0$, pertain to a prismatic beam.

TABLE 1

VALUES OF $(k\;j)\;cos.^2\,\beta$

$100\,p$	0.2	0.6	1.0	1.6	2.0
$\beta = 0$	0.200	0.303	0.359	0.411	0.441
$\beta = 10°$	0.199	0.299	0.352	0.403	0.427
$\beta = 20°$	0.186	0.281	0.329	0.376	0.397
$\beta = 30°$	0.169	0.251	0.294	0.333	0.351
$\beta = 40°$	0.145	0.212	0.246	0.276	0.289

It is seen from the tabular values and equation 15, that M_c as given by equation 15 is always less for $\beta > 0$ than for $\beta = 0$.

In the ordinary theory, given in textbooks on "strength of materials," for homogeneous "beams of equal strength" with variable depth, vertically loaded, it is assumed that at any vertical section the theory for a prismatic beam applies, and that the bending stress at any point of the section acts perpendicular to it. The theory is thus inadequate to express the facts, because it was shown, in the beginning of this paper, that the stress at N, Fig. 1, acts parallel to the face, NN'.

This common theory, if extended to reinforced beams of the type shown by Fig. 1, is on the side of danger, for it would give for any β, the value of M_c from equation 15 corresponding to $\beta = 0$.

The case where $\beta = 0$, $\beta_1 = 0$, leads to the ordinary formulas for a prismatic reinforced beam, for which a valuable working diagram was first given* by Arthur W. French, M. Am. Soc. C. E.

14. *Prismatic Beams with one layer of rods parallel to the surface.* The case is shown in Fig. 8, when $\beta_1 = 0$. Hence we have directly from (10), (11), (12), the following formulas for k, M_s and M_c. To follow the usual notation, put $A_1 = A$ = steel area in section IN for the width b.

$$k = -np + \sqrt{(np)^2 + 2(np)} \quad . \quad . \quad . \quad (17)$$

$$j = 1 - \frac{1}{3}k; \; p = \frac{A}{bd}.$$

$$M_s = f_s A j d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$M_c = \frac{1}{2} f_c (kj) b d^2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Writing, $R_s = f_s p j$, $R_c = \frac{1}{2} f_c k j$,

$$M_s = f_s p j \,.\, b d^2 = R_s b d^2 \quad . \quad . \quad . \quad . \quad (20)$$

$$M_c = \frac{1}{2} f_c k j \,.\, b d^2 = R_c b d^2. \quad . \quad . \quad . \quad (21)$$

For $n = 15$, the values of R_s and R_c were computed for various values of p, f_s and f_c and plotted as ordinates in Fig. 11, the abscissas being the percentage of the reinforcement or $100\,p$. The curves connecting the points, show at a glance the corresponding values of f_c and f_s for a given p and R.

Ex. 1. Thus for $p = 0.01$ or a 1% reinforcement and $R = 108$, we read from the diagram, $f_c = 600$, $f_s = 12{,}700$ lb. per sq. in. If f_c was limited

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* *Trans., Am. Soc. C. E.*, vol. LVI, 1906, p. 362.

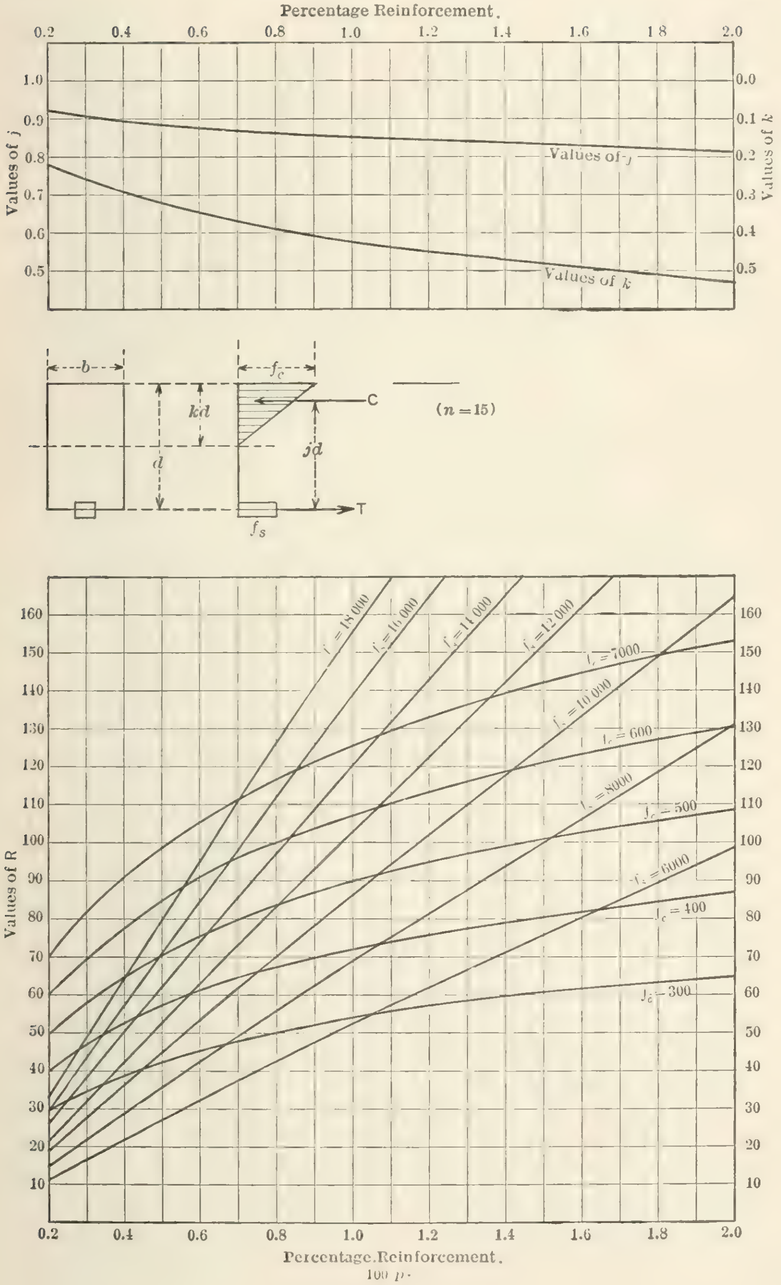

Percentage Reinforcement.
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Values of j
1.0
0.9
0.8
0.7
0.6
0.5
Values of k
0.0
0.1
0.2
0.3
0.4
0.5
Values of j
Values of k
b
f_c
kd
C
(n = 15)
d
jd
T
f_s
Values of R
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
f_s = 18 000
f_s = 16 000
f_s = 14 000
f_s = 12 000
f_c = 700
f_s = 10 000
f_c = 600
f_s = 8000
f_c = 500
f_s = 6000
f_c = 400
f_c = 300
Percentage Reinforcement.
100 p.

FIG. 11

to 600 and f_s to 16,000 lb. per sq. in., the latter value for $p = 0.01$, would correspond to $R = 138$, $f_c > 600$; ∴ the resisting moment is $M_c = R_c\, bd^2 = 108\, bd^2$ and not $M_s = 138\, bd^2$.

Ex. 2. b, d, p, M, given, to find f_c and f_s. Thus let $b = 8''$, $d = 20''$, $p = 0.011$, $M = 400{,}000$ in. lbs. ∴ $R = M/bd^2 = 125$. Find the intersection of a horizontal line through $R = 125$ with a vertical line at $100\,p = 1.1$ and estimate the ordinates, $f_c = 675$, $f_s = 13{,}400$.

Ex. 3. Given $M = 400{,}000$ in. lbs., $f_s = 16{,}000$, $f_c = 600$. At the intersection of the curves for f_s, f_c, the percentage is 0.67 (or $p = 0.0067$) and $R = 96$. ∴ $bd^2 = \dfrac{M}{R} = \dfrac{400{,}000}{96} = 41.70$. If we assume $b = 10''$ ∴ $d^2 = 417$ ∴ $d = 20.4''$ and $A = p.bd = 0.0067 \times 204 = 1.367$ sq. in. For this steel area, f_s and f_c are at the working limits.

15. *Comparison of maximum unit stresses in the concrete as computed by two different methods.* For the toe-beam, Fig. 1, subjected to the upward acting load Q, the external moment about I is $M = Qa$ and on the supposition that f_0 is to be computed as if $\beta = 0$, we have by eq. (15),

$$p_0 = f_0 \sec^2 \beta = \frac{2\,M}{k_0 j_0\, b\, d^2} \sec^2 \beta$$

where k_0, j_0, are to be found from the diagram for k, j, Fig. 9, for $\beta = 0$. This refers to the state of stress shown in Fig. 1 (1). For the state of stress shown in Fig. 1 (2), we have from eq. (15),

$$f_c = \frac{2\,M}{k\,j\,b\,d^2} \sec^2 \beta.$$

Where the values k, j, are to be found from the diagram, Fig. 9, for the assumed value of β. On dividing the first equation by the second, we have,

$$\frac{p_0}{f_c} = \frac{k\,j}{k_0 j_0}$$

The results are given in Table 2 for various values of β and steel percentages, $100\,p = \dfrac{100\,A.}{bd}$

Recalling that p_0 is always in excess of the true value, the excess appreciably increasing with β, the values of f_c, corresponding to System (2) of Fig. 1, are seen to give consistent results and in the right direction. The regularity of the changes in the ratio is noticeable, and the near approach to unity for β = 10 to 20 degrees.

TABLE 2

STEEL PERCENTAGE, 100 p.

$100\,p$ =	0.2	0.6	1.0	2.0
	$\frac{p_0}{f_c}$	$\frac{p_0}{f_c}$	$\frac{p_0}{f_c}$	$\frac{p_0}{f_c}$
β = 10°	1.02	1.02	1.01	1.00
20°	1.05	1.05	1.04	1.02
30°	1.13	1.11	1.09	1.06
40°	1.24	1.20	1.17	1.12

In Fig. 1, $d = I\,N$, b = constant width of section.
In Fig. 1 (1), $k_0\,d = N\,O$, $j_0\,d = D\,I$.
In Fg. 1 (2), $k\,d = N\,O$, $j\,d = D\,I$.

From the experiments on dams, previously referred to, it was found that the vertical unit stress on horizontal planes did not vary uniformly, the stresses near the down-stream face being generally less than corresponds to the law of uniform variation, the difference being practically nothing in the upper part of the dam and increasing as the foundation is approached, where it is comparatively large—considerably over 25%.

From the relation between this stress and that part of it due to flexure alone, we must infer a large proportion of the same excess in the computed over the real value of the stress due to direct bending.

A similar result then would follow for the toe-beam just considered.

In the case of the india-rubber and plasticine dams, the down stream faces made angles of 32° and 41° respectively with the vertical; hence for values of β varying from 30° to 40°, we should expect f_0 and $\therefore$ p_0 for the toe-beam to be in excess of the true stress at N, certainly over 25%. From the ratios in Table 2, it is seen that for such values of β, f_c is in excess and from previous theoretical considerations, this should be so for any value of β greater than zero. However, the difference must be slight when $\beta < 20°$.

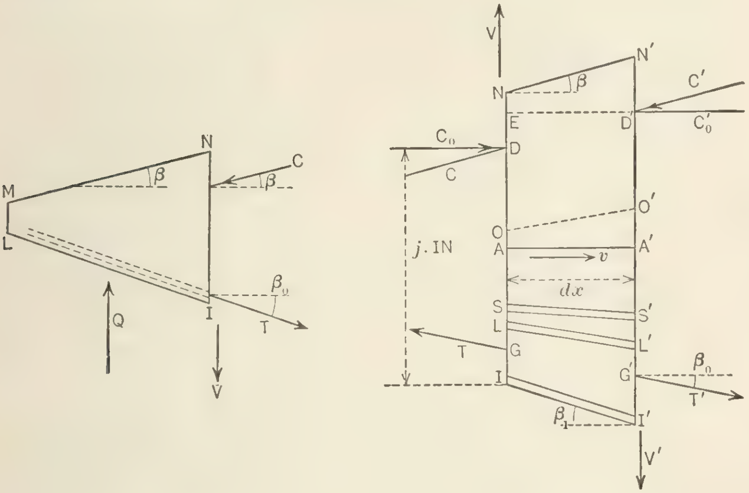
By reference to Table 2, it is seen that p_0 exceeds f_c by from 1% to 5% for $10° < \beta < 20°$ and by from 6% to 24% for $30° < \beta < 40°$, for the various steel percentages. Since the junction of toe and stem is a recognized point of weakness (from the unequal contraction in setting) it may seem desirable to some to use the method which gives a decided excess value. If this is to be done, then the procedure is very simple. Thus if p_0 is assigned a limit of say 650 lb./in.^{2}, then $f_0 = 650 \cos^2 \beta$; with this value and the maximum unit stress allowed in the steel, the diagram for prismatic beams, Fig. 11, can be consulted and the solution effected as usual. As a matter of fact, the maximum unit stress in the concrete of toe-beams, as usually designed, is very small, the steel being the determining factor, so that the final results will not differ very appreciably, whichever method is adopted.

16. *Shear in Wedge-Shaped Reinforced Concrete Beams.* Consider the toe, Fig. 12, subjected to the vertical load Q, the stress C, inclined at the angle β, the tension T in the rod at I inclined at the angle β_0 to the horizontal, and the shear, V, in the section NI, to be carried by the concrete, in addition to the vertical component of C already included. V is thus the shear in the section still to be accounted for after the vertical components of C and T have both been considered. For equilibrium, we must have,

$$V = Q - C \sin \beta - T \sin \beta_0 \quad . \quad . \quad . \quad . \qquad (22)$$

This portion, $LMNI$, of the beam acts on the part of the beam to the right of the section, NI, with forces exactly equal and opposed to C, T, and V.

When the steel is horizontal, $\beta_0 = 0$ and $V = Q - C \sin \beta$. This refers to the case shown in Fig. 1(2). For the case shown in Fig. 1 (1), C is horizontal and $V = Q$. When T, Fig. 12, acts upward, the last term in the first formula changes sign. To make an exact analysis, consider Fig. 13, representing a portion of the beam between the two vertical sections, IN and $I'N'$, dx apart and subjected to the forces, V, V', C and C' (the horizontal components of which are C_0, C_0'), T, the resultant of the tensions in the bars, II', LL', SS', acting at G and inclined at the angle, β_0, to the horizontal and the corresponding resultant, T', at G', supposed to act in the line of T, but opposite in direction. The constant width of the cross-sections will be called b, and the unit shear on a horizontal plane, AA', just below the neutral surface, v. We have,



Figs. 12 and 13

$$I'D' - ID = j\,(I'N' - IN) = j\,dx\,(\tan\beta + \tan\beta_1)$$

$$DE = IE - ID = I'D' - dx\tan\beta_1 - ID = j\,dx\,(\tan\beta + \tan\beta_1) - dx\tan\beta_1 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now, whatever the inclinations assumed of C and C', we have

$$C_0 = T\cos\beta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$C_0' - C_0 = v\,b\,dx \quad . \quad . \quad . \quad . \quad . \quad (25)$$

the latter equation resulting from placing the sum of the horizontal components of the forces acting on $AA'\,N'N$ equal to zero.

Assume the state of stress shown in Fig. 1 (2), so that C and C' are inclined to the normal to IN at the angle β; whence, $C_0 = C\cos\beta$, $C'_0 = C'\cos\beta$, and j is to be found by aid of (3) or the diagram Fig. 9, for the given value of β.

On taking moments of the forces acting on $II'N'N$ about G, we have,

$$C_0' \, (GD + DE) - C_0 \, . \, GD - V' \, . \, dx - C' \sin\beta \, dx = 0$$

$$\therefore \; (C_0' - C_0) \, GD = (V' + C' \sin\beta) \, dx - C_0' \, . \, DE$$

substituting the values of $(C_0' - C_0)$ and DE from (25) and (23) and dividing by dx,

$$vb \, . \, GD = (V' + C' \sin\beta) - C_0' \, [j \, (\tan\beta + \tan\beta_1] - \tan\beta_1$$

On taking the limit as dx approaches zero, so that C_0' is re-replaced by C_0, C' by C and V' by $V = Q - T \sin\beta_0 - C \sin\beta$, from (22), we obtain,

$$vb \, . \, GD = Q + (C_0 \tan\beta_1 - T \sin\beta_0) - j \, C_0 \, (\tan\beta + \tan\beta_1) \quad (26)$$

If we denote by M the moment of the external forces about any point of the section IN, we have, taking moments in turn about G and D,

$$M = C_0 \, . \, GD = T \cos\beta_0 \, . \, GD;$$

$$\therefore \; C_0 = \frac{M}{GD}, \; T = \frac{M}{GD \cos\beta_0}.$$

On substituting these values in (26), it can be written in the form, more convenient for computation,

$$vb \, . GD = Q - \frac{M}{GD}\Big[j \, (\tan\beta + \tan\beta_1) - \tan\beta_1 + \tan\beta_0\Big]. \quad (27)$$

in which Q is the algebraic sum of the loads and reactions (if any) to the left of the section.

For one layer of bars at II', $\beta_0 = \beta_1$, $GD = ID = j \, . \, IN$, $C_0 \tan\beta_1 = T \sin\beta_0$, by (24), (26) and (27) reduce to,

$$vb \, . \, GD = Q - j \, (C \sin\beta + T \sin\beta_1) \quad . \quad . \quad (28)$$

$$vb \, . \, j \, . \, IN = Q - \frac{M}{IN} \, (\tan\beta + \tan\beta_1) \quad . \quad . \quad . \quad (29)$$

The latter is the most convenient form for computing v. The value of v is found from any one of the equations (26)-(29) on dividing by $b \, . \, GD$.

From (28), it is observed, that in the right member, in place of subtracting the sum of the vertical components of C and T from Q, that only j times this amount is subtracted.

When the state of stress represented in Fig. 1 (1) is assumed, C and C' being normal to IN and j, termed now j_0, being computed as if β was zero, the computation proceeds as before and it will be found that the only change in formulas (27) and (29) is that j_0 replaces j, noting too, that GD in (27) is dependent on j_0.

For the common case of one layer of bars, the right member remains the same for both cases, the left members are, $vbj \, . \, IN$ and $v_0 b j_0 \, . \, IN$, for the two cases, where v_0 is the unit shear for the system of stress Fig. 1 (1); whence, $\frac{v}{v_0} = \frac{j_0}{j}$. The latter ratio being very near unity (see Fig. 9), the values v and v_0 are practically the same. The variation of shear over the cross-section will be discussed later.

17. *Bond Stress.* Let $u_1, u_2, u_3, \ldots$, denote the unit bond stresses in the bars, II', LL', SS', . . ., and $\Sigma o_1, \Sigma o_2, \Sigma o_3, \ldots$, the areas per linear inch of surface of the bars, for the width b for the respective bars; then since the shear on the horizontal plane AA', equals the sum of the horizontal components of the bond stresses in the bars,

$$\begin{aligned} b\,v\,dx &= u_1\, II' \cos \beta_1\, \Sigma o_1 + u_2\, LL' \cos \beta_2\, \Sigma o_2 + \ldots \\ &= dx\,(u_1\, \Sigma o_1 + u_2\, \Sigma o_2 + \ldots). \end{aligned}$$

Now, the unit bond stress in any rod is proportional to the unit elongation of the rod, or to its unit stress, which varies with the distance from O (see eq. 6); hence,

$$u_2 = \frac{OL}{OI}\, u_1, \quad u_3 = \frac{OS}{OI}\, u_1, \quad \ldots \qquad (30)$$

Therefore the previous equation can be written, on dividing by dx,

$$u_1 \left[\Sigma o_1 + \frac{OL}{OI}\, \Sigma o_2 + \frac{OS}{OI}\, \Sigma o_3 + \ldots \right] = vb \qquad (31)$$

On writing for vb, its value given by (27),

$$u_1 \left[\Sigma o_1 + \frac{OL}{OI}\, \Sigma o_2 + \frac{OS}{OI}\, \Sigma o_3 + \ldots \right]$$

$$= \frac{1}{GD}\left\{Q - \frac{M}{GD}\left[j\,(tan\ \beta + tan\ \beta_1) - tan\ \beta_1 + tan\ \beta_0\right]\right\} \quad (32)$$

From this equation u_1 can be found and then, by aid of (30), u_2, u_3, . . ., can be computed.

For only one layer of bars at II', vb is given by (29) and, $u_1 = vb \div \Sigma o_1$, or,

$$u_1 = \frac{1}{j\ .\ IN\ \Sigma o_1}\left\{Q - \frac{M}{IN}\,(tan\ \beta + tan\ \beta_1)\right\} \quad .\ \ . \quad (33)$$

If the state of stress, Fig. 1 (1), is assumed, then j is to be replaced by j_0 in the last two equations with a corresponding change in $G\,D$ in (32).

In all the formulas (26)-(33), if T' is directed to the right and upward, *sin* β_0, *tan* β_0, are to be replaced by $(-\sin\ \beta_0)$, $(-\tan\ \beta_0)$, observing too, that for one layer of bars, $\beta_0 = \beta_1$.

For the case of many layers of bars, the value of $G\,D$ can be found as in App., Art. 9 and the direction of T or value of β_0, by drawing the polygon of forces proportional to the tensions in the bars:

$$A_1,\ \frac{OL}{OI}\,A_2,\ \frac{OS}{OI}\,A_3,\ .\ .\ .$$

18. Ex. 1. As an application, let it be proposed to find the unit shear and bond stresses for the heel-slab with the fillet, App., Fig. 7, Art. 9 already considered.

We have given, $Q = 18{,}333$ lbs., $M = 710{,}000$ in. lbs. The tension in the bar which makes the angle $\beta_1 = 43°\ 10'$ with the horizontal $= f_1 A_1$ and that in the horizontal bar $= f_1 \dfrac{O\,L}{OI} A_2 = f_1 \dfrac{11.5}{32} A_2 = 0.361\ A_2$. Hence, since $A_1 = A_2$ in this instance, these tensions are in the ratio 1 : 0.361; therefore at the point of intersection of the two bars on the figure, conceive a force 1 acting along bar 1 and a force 0.361, acting along bar 2 and by construction find their resultant, whose direction is that of T. This resultant is found to make the angle $\beta_0 = 32°\ 05'$ with the horizontal and to cut $N\,I$ at a point G such that $G\,D = 32.1$ inches. These same results can be found analytically.

On substituting in (27), $b = 12$, $G\,D = 32.1$, $j = 0.921$ (Art. 9), $M = 710{,}000$, *tan* $\beta_1 = 0.938$, *tan* $\beta_0 = 0.627$, *tan* $\beta = 0$, we have,

$$12 \times 32.1\ v = 18{,}333 - \frac{710{,}000}{32.1}\,(0.921 \times 0.938 - 0.311)$$

whence, $v = 15.9$ lb./in.^{2}, is the horizontal unit shear between the neutral axis and the horizontal bar.

To compute bond stress, note that both the inclined and horizontal reinforcement consists of ⅞ in. square rods, spaced 8 in.—center to center. Therefore for a breadth $b = 12$ in.,

$$\Sigma o_1 = \Sigma o_2 = \frac{12}{8}(3.5) = 5.25.$$

We have by (31), using the value of v just computed,

$$5.25\, u_1 \left[1 + \frac{11.5}{32} \right] = v\, b = 15.9 \times 12$$

$$\therefore\ u_1 = 26.7 \text{ lb./in.}^2,\quad u_2 = \frac{11.5}{32}\, u_1 = 9.6 \text{ lb.in.}^2,$$

the unit bond stresses in bars 1 and 2.

Ex. 2. For an application of (29) and (33) to the toe of a retaining wall, consider the toe with a fillet Fig. 14, subjected to an upward soil reaction of 10,100 lbs. for $b = 12''$, acting $1.59'$ to the left of the vertical section IN. For a steel percentage about 0.1, from Fig. 9, for $\beta = 45°$, $j = 0.92$; hence assuming $f_s = 16{,}000$ lb./in.^{2}, by (14), we find the corresponding area of metal for $b = 12''$.

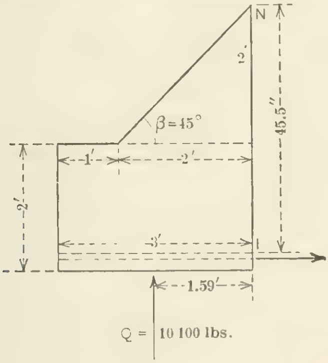


FIG. 14

$$A = \frac{M}{f_s . j\, d} = \frac{10{,}100 \times 1.59 \times 12}{16{,}000 \times 0.92 \times 45.5} = \frac{192{,}900}{670{,}000} = 0.29 \text{ sq. in.}$$

As this is very small, take ⅝″ □ bars, 12″ c to c, the bars being placed 2.5 in. from bottom of toe, giving $A = 0.39$, $\Sigma o_1 = 2.5$.

We have here, $\beta_1 = 0$, $\beta = 45°$,

$$Q - \frac{M}{IN} \tan \beta = 10{,}100 - \frac{192{,}900}{45.5} = 5870$$

Formulas (29 and (33), can be written, putting $IN = d$,

$$u_1 . jd\, \Sigma o_1 = Q - \frac{M}{d} \tan \beta = vb . jd$$

$$\therefore\ u_1 = \frac{5870}{0.92 \times 45.5 \times 2.5} = \frac{5870}{105} = 56 \text{ lb./in.}^2$$

The unit shear v is easily found from the unit bond stress u_1.

$$v = \frac{u_1\, \Sigma o_1}{b} = \frac{56 \times 2.5}{12} = 12 \text{ lb./in.}^2$$

The values of u_1 and v are well within safe limits.

In the solution above, for safety, the friction on the base and the weight of toe were neglected, because the junction of the upper part of the toe with the vertical slab is a recognized point of weakness, from the unequal contraction of the concrete in drying out. Besides the outer surfaces of both toe and stem are in compression. The latter objection does not hold where the heel and stem unite. For abundant security, some designers ignore the fillet of the toe altogether in the computation of u_1, v and A_1, but this plan seems unnecessarily wasteful in steel.

For the commonly recurring case of the *toe*, Fig. 1, where $\beta_1 = 0$, (29) and (33) can be written in the compact form,

$$u_1 \,.\, jd\, \Sigma o_1 = Q - \frac{M}{d} \tan \beta = vb \,.\, jd \quad . \quad . \quad . \quad (34)$$

in which Q and M are the shear and moment at the section IN, due to the external forces.

For a prismatic beam $\beta = 0$

$$\therefore\ u_1 \,.\, jd\, \Sigma o_1 = Q = vb \,.\, jd \quad . \quad . \quad . \quad . \quad (35)$$

19. *Variation in Shear Over Section.* The unit shear q on a horizontal (or vertical) plane at the distance y_1 above O in the case of the toe, Fig. 1 (1), for the state of stress assumed in that figure, can be found by a method similar to that used for prismatic beams, only the rise of N' above N and of O' above O, Fig. 13, must be considered in the limits of the integrations. As the work is long, only the final result will be given. As hitherto, b = constant breadth, d = depth NI, M = moment of external forces about I and k_0, j_0, refer to the state of stress shown in Fig. 1 (1), where C_0 is computed as if the beam were prismatic and of the constant depth d.

The value of q is given by the following equation, in which Q = external shear at section IN:

$$\begin{aligned} q\,(k_0^2 j_0\, b\, d^3) = {} & \\ k_0^2 d\,(Qd - M \tan \beta) & + [2\,(1 - k_0)\, M \tan \beta]\, y_1 \\ & + \left[\frac{3\, M \tan \beta}{d} - Q\right] y_1^2 \quad . \quad . \quad . \quad . \quad (36) \end{aligned}$$

The unit shear at O is found by putting $y_1 = 0$,

$$q \text{ (at } O) = \frac{Q - \frac{M}{d} \tan \beta.}{b \,.\, j_0 d}$$

This is the value of v, found from (29), on changing j to j_0 to pertain to the state of stress assumed.

At N, the shear is found by putting $y_1 = k_0 d$.

$$q \text{ (at } N) = \frac{2 M \tan \beta}{k_0 j_0 bd^2} = f_0 \tan \beta,$$

as can be proved independently by considering the conditions of equilibrium of an infinitesimal wedge at N.

Another check is afforded by taking $\beta = 0$, whence

$$q = \frac{Q \, (k_0^2 d^2 - y_1^2)}{k_0^2 j_0 \, b \, d^3}$$

which is easily proved independently.

In Fig. 1 (1), Q may represent the soil reaction (friction omitted), which acts at the distance a to the left of IN; whence $M = Q a$.

On dividing eq. (34) by $(k_0^2 j_0 \, bd^3)$, it can be put in the form,

$$q = l + m \, y_1 + n \, y_1^2,$$

where q and y_1 are the variables, l and m are both positive, though n may be positive or negative.

When $n = 0$ this is the equation of a straight line, q increasing with y_1. When n is not zero it is the equation of a parabola. Let d' = vertical depth down to the steel, of the left or truncated end of the toe and suppose Q to act approximately at the center of the base, corresponding to a uniformly distributed soil reaction; then it can be shown that $n > 0$, $= 0$, < 0, according as $d' < \frac{d}{3}, = \frac{d}{3}$ or $> \frac{d}{3}$ and that q increases with y_1 for $d' < \frac{d}{3}$ or $d' = \frac{d}{3}$, but for $d' > \frac{d}{3}$, q may first increase and then decrease or it may increase throughout or decrease throughout, as y_1 varies from 0 to ON. For all cases, the value of q at N is $f_0 \tan \beta$.

It may be well to remark that the unit shear on a plane inclined to the horizontal may be greater than the shear on a horizontal plane, since there is generally a vertical component of stress to be considered in wedge-shaped beams.

The general formula above refers to the state of stress shown in Fig. 1 (1). If the state of stress of Fig. 1 (2) is assumed the same formula (34) is derived, except that k_0, j_0, are to be replaced by k, j, which agrees with previous results for $y_1 = 0$.

Reasons have been given in App., Art. 3, for the belief that the linear law of variation of stress shown in Fig. 1, (1) or (2), is only approximately true. It follows that the variation in shear, as given by (36), which is based on this law, is only approximately true, the actual shear at the surface being less than the equation indicates.

20. *Spacing of Bars.* It is a great convenience to have at hand tables, giving the spacing of round and square bars for various areas of metal for 12″ width, such as is given in Marsh and Dunn's "Reinforced Concrete Manual," pp. 238-241; but lacking such, a little computation is necessary.

Thus, for one layer of bars, let A = cross-section of metal in width b, x = spacing of bars or distance from center to center of bars, each one having the cross-sectional area a. Then conceiving the metal in a continuous sheet,

$$\frac{A}{b} = \frac{a}{x} = \text{area in 1 unit width.}$$

Given three of the quantities A, a, b, x, the other can be found from this equation.

Thus if A = 1.69 sq. in., for b = 12″ and we wish the spacing of ¾″ square bars $\therefore a = \frac{9}{16}$ sq. in.

$$\therefore x = \frac{a\,b}{A} = \frac{\frac{9}{16} \times 12}{1.69} = 4 \text{ inches spacing.}$$

Similarly in bond stress, using previous notations, $\frac{\Sigma o}{b} = \frac{o}{x} =$ area of surface of bars per linear inch for unit width.

Thus the spacing, x inches for a bond stress, $u_1 = 80$ lbs. sq. in., is, $x = \frac{bo}{\Sigma o}$; or since by Art. 18, Ex. 2 for one layer of bars, $\frac{1}{\Sigma o} = \frac{u_1}{vb} \therefore x = \frac{ou_1}{v} = 80\frac{o}{v}$, where v = unit shear computed from (29).

For prismatic beams, by (35), $\frac{1}{\Sigma_o} = \frac{u_1 jd}{Q} \therefore x = \frac{80\, b . jd . o.}{Q}$

21. *Compressive Stress in Concrete Due to Bending a Reinforcing Rod Under a Tension T.* Suppose a square rod of

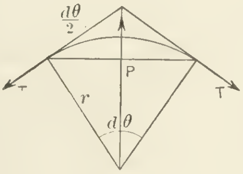


FIG. 15

diameter b to be bent to a circular arc of radius r. Consider a small portion of this arc, with a central angle $d\theta$ and length, $rd\theta$ and let the unit stress in the bent rod be f, whence the total stress in it $= T = fb^2$. See Fig. 15. The bent rod presses the concrete, the area of contact being $b\,r\,d\,\theta$ and if p = unit compression, the total resisting force of the concrete is, $P = p\,b\,r\,d\theta$ ($d\theta$ being very small). The forces T, T, P of the figure are in equilibrium, hence balancing components parallel to P,

$$p\,b\,r\,d\theta = 2\,T\,sin\frac{d\theta}{2} = T\,d\theta = f\,b^2 . d\theta$$

Here, the sine has been replaced by the arc, since $d\theta$ can be made as near zero as one pleases.

Hence the formula,

$$p\,r = f\,b \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

If we take $f =$ 16,000 lbs. sq. in. and $r = 25\,b$,

$$p = \frac{16{,}000}{25} = 640 \text{ lbs. sq. in.,}$$

which is about the working limit of concrete in compression. The radius should not be less than 25 diameters of the square rod and preferably more. Sharp bends should always be avoided or rupture may ensue.

If the rod is round, of diameter b, let the effective area be taken roughly, the same as before.

$$\therefore\ p\,b\,r\,d\theta = T\,d\theta = f\,\pi\,\frac{b^2}{4}\,d\theta$$

$$p\,r = 0.785\,f\,b \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If $r = 20\,b, f = 16{,}000, p = 628$; so that about 20 diameters of the rod is the minimum radius that should be used in the case of bent round rods. Similarly, when a straight rod under a stress of 16,000 lbs. sq. in., is fastened by screw and nut to a flat plate, bearing against concrete, the bearing area of the plate should be at least, $\frac{16{,}000}{650} = 25$ times that of the steel.

22. *Length of Bar to Embed in Concrete.* Let us use a working bond stress for plain bars of 80 lbs. sq. in., which is about one-third of the ultimate.

With a square bar of diameter d, let $l =$ length to embed to develop the required bond strength, corresponding to a tensile unit stress of 16,000 lbs. sq. in. The area of the surface is $4\,d\,l$;

$$\therefore\ 4\,d\,l \times 80 = 16{,}000\,d^2 \ \therefore\ l = 50\,d.$$

For round bars of diameter d, we have, similarly, $\pi d l \times 80 = 16{,}000\,\pi \frac{d^2}{4} \quad \therefore \; l = 50\,d$.

Hence for plain round or square bars, an embedment of 50 diameters is safe. For deformed bars, allowing a bond stress of 100 lbs. sq. in., an embedment of 40 diameters is allowed. This embedment will likewise suffice for plain round or square bars with hooked ends, bent 180° to a radius of 3 diameters with a short length of bar beyond the bend.

23. *Working Stresses.* Although the elastic limits of soft, mild and hard steel may average 33,000, 37,000 and 55,000 lbs. sq. in., respectively, yet the modulus of elasticity of either kind of steel is about 30,000,000 lbs. sq. in., which is 10 to 15 times that for concrete. It follows, if the steel in a beam is stressed to 3000 lbs. sq. in., since the surrounding concrete stretches as much as the steel, the stress in the concrete is $\frac{3000}{15} = 200$ lbs. sq. in., when the modulus for the steel is 15 times that for the concrete.

As this is about the limit for the tensile stress of the concrete, as the stress in the steel increases, invisible cracks occur which become visible at 10,000 to 16,000 lbs. sq. in. stress in the steel,* so that it is regarded undesirable, for any kind of steel, to exceed the limit 16,000. We thus gain no advantage, within working limits, in using the hard steel, in spite of the fact, that its elastic limit and breaking strength are so much greater than for mild steel.

As giving a résumé of the practice of the present day, the reader is referred to the "Report of Special Committee on Concrete and Reinforced Concrete," published in Proceedings Am. Soc. C. E. for Feb., 1913 (or Trans., 1914).

A few of the recommendations will be given here that apply more particularly to the design of the slabs of reinforced concrete retaining walls.

---

* See Turneaure & Maurer's "Reinforced Concrete," p. 40.

The committee advises that the following values of the ultimate compressive strength of concrete, in pounds per square inch, should be the maximum values allowed for static loads:

Aggregate	1:1:2	1:1½:3	1:2:4	1:2½:5	1:3:6
Granite, trap rock	3,300	2,800	2,200	1,800	1,400
Gravel, hard limestone and hard sandstone	3,000	2,500	2,000	1,600	1,300
Soft limestone and sandstone	2,200	1.800	1,500	1,200	1,000
Cinders	800	700	600	500	400

"The extreme fiber stress of a beam, calculated on the assumption of a constant modulus of elasticity under working stresses, may be allowed to reach 32.5% of the compressive strength." For a 2000 lbs. per sq. in. ultimate strength, this would limit f_c to 650 lbs. sq. in.

In beams, where the computed maximum shearing stress is used as the means of measuring the diagonal tension stress, the maximum shearing stress may be 2% of the ultimate. This refers to prismatic beams without web reinforcement.* For the 2000 ultimate, the limiting value of v is 2% of 2000 = 40 lbs. sq. in.

For concrete in shear not combined with tension or compression-punching shear—6% of the ultimate or 120 lbs. sq. in. for the 2000 ultimate may be used.

The unit bond stress between concrete and plain reinforcing bars, is allowed to reach 4% of the ultimate or 80 lbs. sq. in. for an ultimate of 2000 lbs. sq. in.

The committee does not give figures as to deformed bars, which are said to have a bond stress (for the 2000 ultimate) of 80-150 lbs. sq. in.; neither do they refer to the usual increase in bond stress to 100 lbs. sq. in. for plain bars hooked properly at the ends.

---

* The diagonal tension stress is a function of the shear and moment, though experiments on prismatic beams seem to indicate that shear is the chief factor, which fact is provisionally assumed in dealing with the toes and heels of the walls of Chap. IV, for which experimental data are lacking.

For wedge-shaped beams, the unit shear at the neutral axis is smaller than for the enclosing prismatic beam, and it is possible that shear is not to the same extent "to be used as the means of measuring diagonal tension." Perhaps a value of the unit shear intermediate between that for the toe and prismatic beam may be more suitable for use with the assigned limit to the unit shear, as to which, at present, the best judgment of the engineer must be exercised.

The modulus of elasticity of concrete has a wide range. If n = ratio of modulus of steel to that of concrete and s = ultimate compressive stress in concrete, then it is recommended in computations for locating the neutral axis or finding the resisting moment in beams, to take,

$n = 15$, when s is 2200 lbs. sq. in. or less,
$n = 12$, when $s > 2200$ but < 2900 lbs. sq. in.,
$n = 10$, when s is greater than 2900 lbs. sq. in.

The lateral spacing of parallel bars should not be less than three diameters, from center to center, nor should the distance from the side of the beam to the center of the nearest bar be less than two diameters.

Reinforcement for shrinkage and temperature stresses should not generally be less than 1/3 of 1%.

In the case of continuous beams and slabs, the bending moment at the center and at support for interior spans shall be taken at $(1/12)\ wl^2$, where w represents the load per linear foot and l the length in feet, of the span.

APPENDIX II

Experiments on Retaining Walls

The following data, referring to model retaining walls at the limit of stability, are taken from a paper by the author on "Experiments on Retaining Walls and Pressures on Tunnels,"* to which the reader is referred for many additional experiments and a much fuller discussion.

As hitherto, w and w' represent the weight in pounds per cubic foot of filling and wall respectively, φ = angle of repose of filling, φ' = angle of friction of filling on wall. For the vertical rectangular walls, Figs. 16, 17, 18, 19, h = height, t = thickness of wall in feet. The earth thrust against the wall is supposed to act at one-third the height from the base of the wall to the surface of the filling.

In the first four designs, where the surface of the filling was horizontal, the centers of pressure on the base or base produced, were found, (1) by the Rankine theory, in which the earth thrust is supposed to act horizontally and (2) where this thrust is supposed to be inclined at the angle φ' below the normal to the wall. On combining these thrusts with the weight of wall, as usual, the resultant is found to strike the base or the base produced, at R in the first case (Rankine's theory) and at I by the second method. The earth thrust in case (1) was computed by Rankine's formula (Art. 48), in case (2), by the formulas of Art. 44, which include the influence of the wall friction.

For brevity let,

q = distance from center of pressure on the plane of the base to the outer toe, divided by t, using the theory that rationally includes the whole of the wall friction;

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* *Trans. Am. Soc. C. E.*, vol. LXXII, p. 403 (1911).

q_0 = same, according to the Rankine theory. q and q_0 will be taken as positive when the resultant on the base cuts it within the base; otherwise negative.

Fig. 16 represents Lieut. Hope's wall of bricks laid in wet sand, backed by sand level with its top: $h = 10$, $t = 1.92$, $w = 95.5$, $w' = 100$, $\varphi = \varphi' = 36° 53'$. The wall was 20 ft. long. Ignoring any leaning, $q = + 0.04$, $q_0 = - 0.58$; but if we reckon the probable overhang at the moment of failure as 4 in., it is found that $q = - 0.02$.

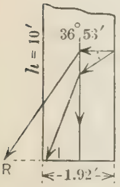

FIG. 16

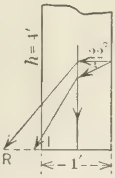

FIG. 17

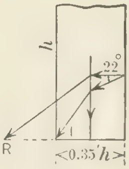

FIG. 18

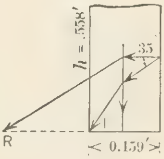

FIG. 19

Fig. 17 shows Baker's wall of pitch-pine blocks, backed by macadam screenings, the level surface of which was 0.25 ft. below the top of the wall: h = 4, $t = 1$, $w = 101$, $w' = 46$, $\varphi' = 39°$, φ' (assumed) $= 22°$, $q = - 0.06$, $q_0 = - 0.55$.

Trautwine's experimental wooden wall is shown in Fig. 18. Only the ratio of base to height, 0.35 was given, but the wall was probably 6 inches in height and it was backed by sand: $w = 89$, $w' = 28$, $\varphi = 33° 42'$, φ' (assumed) $= 22°$, $q = - 0.03$, $q_0 = - 0.75$.

The wall of Curie, Fig. 19, was of wood coated on the back by sand with sand filling, so that $\varphi = \varphi' = 35°$. Also $h = 0.558$ ft., $t = 0.159$ ft. $w/w' = 3.29$, $q = 0.02$, $q_0 = - 1.32$.

It will be observed in these four walls, that R is far outside the base, indicating strikingly the inaccuracy of the Rankine method when applied to such walls, with the free surface of the earth horizontal.

In the wall, Fig. 20, of Curie's, the retaining board AB, 1 meter square, was coated with sand and in the first experiment, the angle BAD with the vertical AD, was 27° 30'. The "limiting plane" (Arts. 27-31), for the earth surface BE level, bisects the angle between the vertical and the natural slope and thus for the given $\varphi = 33^\circ 30'$, it makes an angle with the vertical of $\frac{1}{2}(90^\circ - 33^\circ 30') = 28^\circ 15'$, which is greater than BAD. Hence strictly the construction of Art. 40, Fig. 20 applies,

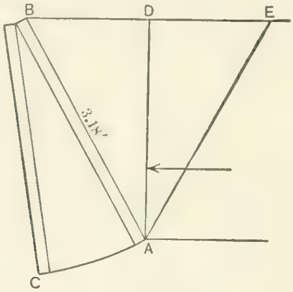

FIG. 20

using $\varphi' = \varphi$; but as AB so nearly coincides with the limiting plane, it is sufficiently near to use the Rankine method and compute the horizontal earth thrust on the vertical plane AD by the formula of Art. 48 and combine this with the weight of sand BAD and the weight of frame ABC to find the resultant on the base AC. It is found to cut it 0.11 of its width from the outer toe $C \therefore q = 0.11$.

In the second experiment, the angle BAD was 55°, $\varphi = 33^\circ 30'$ and $AC = 0.548$ meters. Since AB is below the limiting plane, by Art. 29, the same method is pursued, giving $q = 0.02$. The third experiment was on a smaller retaining board. Here $AB = 0.2$ meter, $BAD = 55^\circ$, $\varphi = \varphi' = 35^\circ$ and $q = 0.06$. Thus in the last two experiments, the resultant on the base passed practically through C.

In Fig. 21 is shown a surcharged wall of Curie's, having $h = 2.952$ ft., $t = 0.755$ ft., the level upper surface of the surcharge being 4.26 ft. above the top of the wall. The surcharge extended over the wall at the angle of repose for the damp sand filling, $\varphi = 45^\circ$. Experiment gave $\varphi' = 35^\circ$. The wall was of brick in Portland cement and $w/w' = 0.62$. The thrust on AF was found by the graphical method of Art. 39 and it was assumed to act (see Art. 43) 0.364 AF above A. On combining the thrust with the weight of the wall, the resultant is found to cut the base 0.02 of its width outside the base, so that $q = -0.02$.

With the exception of Trautwine's wall, the filling was contained in a box, so that there was friction of the filling on the sides of the box, which has been neglected in the investigation above. When the wall is long, this can have but little influence on the results, but where the length of wall is not much greater than its height, the side friction becomes appreciable. By aid of special experiments, Leygue found that for walls having a length twice the height, that the true thrust was diminished about 5% from the side friction for level-topped earth and as much as 15% for the surface sloping at the angle of repose.

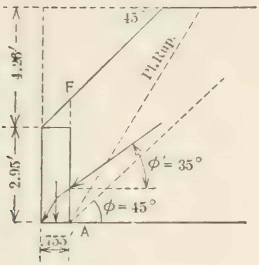

FIG. 21

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